

Infinitesimal deformation of ultrametric differential equations

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ABSTRACT

This paper is devoted to provide an equivalence between a class of differential equations and a class of σ -modules called Taylor admissible. We obtain this result under the assumption that the automorphism σ satisfies some conditions called infinitesimality and non degeneracy. We perform our computations in the framework of Berkovich spaces. We give then three applications. The first one about the action of the group $\Gamma_K := \text{Gal}(K_\infty/K)$ appearing in the Fontaine's theory of (φ, Γ_K) -modules, and to the functor N_{dR} of L.Berger. The second application concerns the theory of the so called finite differences equations: we generalize the results of [ADV04] and [Pul07] to the automorphisms of the form $f(T) \mapsto f(qT + h)$. As a third application we use the functional equation of the Morita's p -adic Gamma function to prove that it is solution of a differential equation. We relate then the radius of convergence of this particular equation to the absolute value of the values at positive integers of some Kubota-Leopoldt's p -adic L -functions. We find finally a family of new congruences between these values which are close to that of [Was98] and [Bar83].

Version of September 14, 2008

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2000 Mathematics Subject Classification Primary 12h25; Secondary 12h05; 12h99; 11S40; 11S80; 11M99

Keywords: p -adic differential equations, p -adic difference equation, σ -modules, p -adic L -functions, p -adic q -difference equation, Deformation, unipotent, p -adic local monodromy theorem, canonical extension, (ϕ, Γ) -modules

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Introduction

In [ADV04] one proves the existence of a q -confluence functor which an equivalence of categories associating to a q -difference equation a differential equation (cf. also [Pul07]). We recall that a q -difference equation is a finite free module together with a semi-linear action of the automorphism $\sigma_q : f(T) \mapsto f(qT)$. In this paper we show the analogous theorem for a more general class

of automorphisms called *infinitesimal*. Roughly speaking an automorphism is infinitesimal if it is “*sufficiently close to the identity*”. More precisely if Σ is a family of infinitesimal automorphisms of the base space, we say that a differential equation is Σ -compatible if, for all point y of the base space, the disk of convergence of its Taylor solution at y is globally stable under the action of Σ . (Notice that every differential equation is Σ -compatible with respect to some infinitesimal family Σ .) For Σ -admissible equations we prove the existence of a canonical semi-linear action of Σ on the differential module, which becomes then a Σ -module. The canonical action of Σ is obtained by “pull-back” of the stratification attached to the differential equation (cf. Section 5.3.3). Roughly speaking this action is determined by the fact that the Taylor solutions of the differential equation are at the same time solution of the underling Σ -module. Morphisms of differential equations commutes also with the action of Σ . The Σ -deformation functor Def_Σ associates then to a Σ -admissible differential equation its canonical underling Σ -module and is the identity on the morphisms (cf. Section 5.3.3). If moreover the family Σ satisfies certain *non degeneracy* conditions (satisfied by the most part of families), then the Σ -deformation functor is *fully faithful* (cf. Section 5.4). In this case the data of the differential equation is completely equivalent to the data of the corresponding Σ -module. We then call the objects in the essential image of the Σ -deformation functor *Taylor admissible Σ -modules*, and a quasi inverse of Σ -deformation functor is called *Σ -Confluence functor* Conf_Σ . We perform this description for sub-affinoids of \mathbb{A}^1 in the framework of Berkovich spaces, and examine many other situations. Basic results are made by the understanding of the behavior of the Generic radius of convergence of the equations in the Berkovich space of the affinoid along the lines of [BV07] (cf. Section 2.7). In a second moment we focus on the Taylor admissible Σ -modules over the Robba ring. We prove the analogous of the Katz-Matsuda canonical extension functor (cf. Section 6.5), and the quasi unipotence of Taylor admissible Σ -modules (cf. Section 6.6). We obtain these results by deformation of the same results for differential equations. The main point here is to compare the Taylor solutions and étale solutions (i.e. solutions lying in an étale extension of the Robba ring).

In the second part we apply then this theory to three cases: to the theory of (φ, Γ_K) -modules (cf. Section 7), to the theory of finite difference equations (cf. Section 8), and finally we give an application to the Morita’s p -adic Gamma function and to values of Kubota-Leopoldt’s p -adic L -functions (cf. Section 9). As we will see the language introduced in the present paper permits to interpret (a part of) the theory of L.Berger and the theory of q -difference equations introduced by Y.André and L.Di Vizio. In the theory of confluence of q -difference equations, one obtains a differential equation from the action of the q -difference operator σ_q . On the other hand L.Berger is able to obtain a differential equation from the (φ, Γ_K) -module attached to a de Rham representation. From this point of view the theory of L.Berger can be seen as a sort of *confluence theory* of (φ, Γ_K) -modules over the Robba ring (cf. Theorem 0.1).

Application to the theory of Fontaine’s (φ, Γ_K) -modules

Let K be a finite extension of \mathbb{Q}_p and $K_n := K(\mu_{p^n})$, where μ_{p^n} is the group of p^n -th roots of unity. In this context a subgroup of Γ_K of finite index acts infinitesimally on the Robba ring, and we can apply the above theory with $\Sigma := \Gamma_K$. Let us firstly recall that L.Berger was able in [Ber02] to prove that there is a faithful functor N_{dR} associating to a de Rham representation of $\mathcal{G}_K := \text{Gal}(K^{\text{alg}}/K)$ a p -adic differential equation over the Robba ring. This functor have been very fruitful. It is possible to translate certain problems concerning the starting representation V in terms of the p -adic differential equation $N_{dR}(V)$. Namely, thanks to the fundamental work of L.Berger (cf. [Ber02]), one knows that V is potentially semi-stable if and only if $N_{dR}(V)$ is quasi unipotent. Moreover $N_{dR}(V)$ is unipotent (resp. trivial) if and only if the restriction of V to $\mathcal{G}_{K_n} := \text{Gal}(K^{\text{alg}}/K_n)$ is semi-stable (resp. cristalline). These facts leaded L.Berger to prove that V satisfies the Fontaine conjecture

“*de Rham = potentially semi-stable*” if and only if $N_{dR}(V)$ satisfies the Crew’s conjecture “*every p -adic differential equation admitting a Frobenius Structure is quasi unipotent*”, which have been proved by Y.André, Z.Mebkhout, and K.Kedlaya (cf. [And02],[Meb02],[Ked04]). Another interesting result in this context is due to A.Marmora (cf. [Mar04]) who proved that the p -adic irregularity of the differential equation $N_{dR}(V)$ is equal to the Swan conductor of the restriction $V|_{\mathcal{G}_{K_n}}$, for n sufficiently large. In fact the sequence of Swan conductors $\text{swan}(V|_{\mathcal{G}_{K_n}})$ become constant, for n sufficiently large. It is today a belief that one can expect to read into the differential equation $N_{dR}(V)$ every problem of V whose nature is “*potential*”. In particular one expects that every differential invariant of $N_{dR}(V)$ corresponds to a Galois invariant of $V|_{\mathcal{G}_{K_n}}$, for n sufficiently large.

In this part of the paper we prove that the above correspondence of invariants reflects actually the existence of a genuine *equivalence of categories*. For this let us firstly recall some results recently proved by L.Berger. In [Ber07] Berger is able to prove that the Colmez-Cherbonnier’s equivalence between p -adic representations and (φ, Γ_K) -modules sends the sub-category of de Rham representations into a certain subcategory that he is able to characterize, whose definition is too technical to be exposed here. He obtains then an equivalence between de Rham representations of $\mathcal{G}_K := \text{Gal}(K^{\text{alg}}/K)$ and a certain category of (φ, Γ_K) -modules over the Robba ring, here denoted by $(\varphi, \Gamma_K) - \text{Mod}(\mathcal{R}_{F'})^{\text{ét}, \text{LT}}$ (cf. [Ber07], and Section 7.1). Berger is then able to factorize the functor N_{dR} through this category (we refer to Section 7.1 for more details about the notations):

$$N_{dR} : \text{Rep}_{dR}(\mathcal{G}_K) \xrightarrow[\sim]{D^\dagger} (\varphi, \Gamma_K) - \text{Mod}(\mathcal{R}_{F'})^{\text{ét}, \text{LT}} \xrightarrow{C_{\Gamma_K}} (\varphi, \nabla) - \text{Mod}(\mathcal{R}_{F'})^{\text{ét}} \rightarrow d - \text{Mod}(\mathcal{R}_{F'})^{(\varphi)}$$

where $(\varphi, \nabla) - \text{Mod}(\mathcal{R}_{F'})^{\text{ét}}$ is the category of (φ, ∇) -modules which are étales in the sense of [Ber07, Def. IV.1.2],¹ and $d - \text{Mod}(\mathcal{R}_{F'})^{(\varphi)}$ is the category of differential equations admitting an unspecified action of φ (cf. Section 6.3.2).

Motivated by the above correspondence between Galois and differential invariants we are induced to consider the categories

$$\begin{aligned} \text{Germ Rep}_{dR}(\mathcal{G}_K) &:= \varinjlim_n \text{Rep}_{dR}(\mathcal{G}_{K_n}), \\ \text{Germ } (\varphi, \Gamma_K) - \text{Mod}(\mathcal{R}_{F'})^{\text{ét}, \text{LT}} &:= \varinjlim_n (\varphi, \Gamma_{K_n}) - \text{Mod}(\mathcal{R}_{F'})^{\text{ét}, \text{LT}}. \end{aligned}$$

Here the inductive limits are considered with respect the natural restrictions from K_n to K_{n+1} . We have again the functors

$$\text{Germ Rep}_{dR}(\mathcal{G}_K) \xrightarrow[\sim]{D^\dagger} \text{Germ}(\varphi, \Gamma_K) - \text{Mod}(\mathcal{R}_{F'})^{\text{ét}, \text{LT}} \xrightarrow{C_{\Gamma_K}} (\varphi, \nabla) - \text{Mod}(\mathcal{R}_{F'})^{\text{ét}} \rightarrow d - \text{Mod}(\mathcal{R}_{F'})^{(\varphi)}.$$

Now, by the above theory of deformation, we are able to prove the existence of an equivalence

$$\text{Def}_{\Gamma_K} : (\varphi, \nabla) - \text{Mod}(\mathcal{R}_{F'})^{\text{ét}} \xrightarrow{\sim} \text{Germ}(\varphi, \Gamma_K) - \text{Mod}(\mathcal{R}_{F'})^{\text{ét}, \text{adm}}$$

where the last category is the full subcategory of $\text{Germ}(\varphi, \Gamma_K) - \text{Mod}(\mathcal{R}_{F'})^{\text{ét}}$ formed by admissible objects (see Section 7.4 for more details).

THEOREM 0.1 (cf. Theorems 7.13 and 7.14). *We have the inclusion*

$$\text{Germ}(\varphi, \Gamma_K) - \text{Mod}(\mathcal{R}_{F'})^{\text{ét}, \text{LT}} \subseteq \text{Germ}(\varphi, \Gamma_K) - \text{Mod}(\mathcal{R}_{F'})^{\text{ét}, \text{adm}} \quad (0.1)$$

and the functor C_{Γ_K} is a quasi inverse of Def_{Γ_K} (and hence coincides with the Γ_K -confluence functor Conf_{Γ_K}). In particular $\text{Germ Rep}_{dR}(\mathcal{G}_K)$ is quasi isomorphic to a fully faithful subcategory of $(\varphi, \nabla) - \text{Mod}(\mathcal{R}_{F'})^{\text{ét}}$.

¹Here the upper index LT means that the Lie algebra of Γ_K acts *locally trivially* in the sense of [Ber07, Def. III.1.2]. The notations of the above categories differs from [Ber07], and are “non standard”.

It is unknown whether the inclusion (0.1) is an equality. If it is not the case, then the above theorem induces to suspect the existence of an extension of the functor N_{dR} to a larger category.

Finite difference equations

Let K be a general complete valued ultrametric field of characteristic 0. In Section 8 we apply the deformation theory to the so called *finite difference equations* i.e. the automorphism σ is of the type $\sigma_{q,h}(f(T)) := f(qT + h)$, $q \in K^\times$, $h \in K$. We generalize the previous papers on this subject [ADV04], [DV04], [And04], [Pul07]. In particular this section generalize the results of [Pul07] in three aspects: firstly we generalize the theory of q -difference equations to the case of automorphisms of the type $\sigma_{q,h}$; secondly we generalize the theory from the p -adic case to the general ultrametric case; finally we simplify and generalize many technical assumptions. In this context we have a richer situation because we dispose of a so called *twisted Taylor formula* for $\sigma_{q,h}$ -difference modules (cf. formula (8.11)). This permits to express explicitly the infinitesimality and the non degeneracy of $\sigma_{q,h}$ in terms of q and h . It permits also to give a criterion for the Taylor admissibility of Σ -modules which is given in terms of the powers of the (q, h) -twisted derivation $\Delta_{q,h} := \frac{\sigma_{q,h}-1}{(q-1)T+h}$. The existence of such a twisted Taylor formula seems to be a peculiarity of the automorphism $\sigma_{q,h}$. In fact for a general automorphism σ the existence of such a twisted Taylor formula is very improbable because the twisted σ -derivation $\Delta_\sigma := \frac{\sigma-1}{\sigma(T)-T}$ presents denominators and hence singularities. One of the results we obtain is that the category of solvable differential equations over the Robba ring is equivalent to that of Taylor admissible semilinear representations of μ_{p^∞} , where a p^m th root of unity ξ acts on the Robba ring by $f(T) \mapsto f(\xi T)$ (cf. Corollary 8.8). Composing this result with that of [And02], we find that the category of linear representations of the Inertia $\mathcal{I}_{k^{\text{alg}}((t))}$ with coefficients in K^{alg} , is equivalent to that of finite free modules over the Robba ring $\mathcal{R}_{K^{\text{alg}}}$ together with an unspecified Frobenius and a *Taylor admissible* semi-linear action action of μ_{p^∞} .

Morita's p -adic Gamma function and values at positive integers of Kubota-Leopoldt's p -adic L -functions

In Section 9 we apply the above theory of finite difference equations to the case of the particular difference equation satisfied by the Morita's p -adic Gamma function. We interpret the functional equation of $\Gamma_p(T)$ as a finite difference equation, and from this fact we are able to deduce, by the above theory, that the Gamma function is solution of a rank one differential equation $\Gamma_p^0(T)' = g_0(T)\Gamma_p^0(T)$ where g_0 is an analytic function on the open unit disk $D^-(0, 1)$ (here Γ_p^0 denotes the Taylor expansion of the Gamma function at $T = 0$). We compute then completely the Radius of convergence of this differential equation which closely related to the Newton polygon of $g_0(T)$. We perform our constructions by expressing the differential equation as “limit” of the family of finite difference equations

$$\{ \Gamma_p(x + p^n) = A(1, p^n; T) \cdot \Gamma_p(T) \}_{n \geq 1}, \quad (0.2)$$

and computing step by step the Radius of the p^n th equation. In our knowledge, this seems to be the first time in which it results necessary to consider a confluent family of difference equations in order to study a differential equation. In this sense, this seems to be the first case in which the theory of p -adic confluence of difference equation allow to prove new results for differential equations.

Values at positive integers of p -adic L functions. During the accomplishment of this work we benefited of discussions with D.Barsky who suggested us to apply the above computations about the p -adic Gamma function to the formula

$$\log(\Gamma_p^0(T)) = \lambda_0 T + \sum_{m \geq 1} \frac{L_p(2m+1, \omega_p^{2m})}{2m+1} T^{2m+1}, \quad |T| \leq |p|, \quad (0.3)$$

where $L_p(1+2m, \omega_p^{2m})$ is the value at $1+2m$ of the Kubota-Leopoldt's p -adic L -function corresponding to the Dirichlet character ω_p^{2m} , where ω_p is the Teichmüller character. Indeed by considering the derivative of this expression we find

$$g_0(T) = \lambda_0 + \sum_{m \geq 1} L_p(1+2m, \omega_p^{2m}) T^{2m}. \quad (0.4)$$

The knowledge of the Newton polygon of $g_0(T)$ gives a bound on the absolute value of $L_p(1+2m, \omega_p^{2m})$. These values together with the Newton polygon of $g_0(T)$ are *intimately related* to the radius of convergence of the differential equation $\Gamma_p^0(T)' = g_0(T)\Gamma_p^0(T)$.

From the fact that $\Gamma_p^0(T)$ is simultaneously solution of every equation of the family (0.2) and of the “limit” differential equation, we deduce a family of congruences satisfied by the values $L_p(1+2m, \omega_p^{2m})$. Similar congruences were already known by [Was98] and [Bar83]. The main innovation here, with respect to the previous authors, is the fact that these new congruences can be understood in terms of the above theory. In our knowledge, this seems to be the first time that techniques of p -adic difference and differential equations are used to study values of p -adic L -functions.

Comments about Berkovich spaces

In this paper the language of Berkovich spaces displayed all its efficaciousness. We introduce in fact the notion of *maximal Skeleton* and of *critical points* of the Berkovich space of a bounded sub- K -affinoid of \mathbb{A}^1 using its natural partial order (cf. Def. 2.8 and 2.9). This notion permits to test on the *finite set of critical points* some properties involving all the points, like for example “for all Ω/K and all $y_0 \in X(\Omega)$, the norm of $|f(y_0)|_\Omega$ is smaller than ...” or again “for all Ω/K and all $y_0 \in X(\Omega)$, the radius of $Y(x, y)$ at $y = y_0$ is greater than ...”. In particular we approach, in Section 2.7, the study of the convergence locus of a two-variable function converging on a tubular neighborhood of the diagonal of $X \times X$. Indeed *the convergence locus of such a function is usually larger than the tubular neighborhood itself*. We give a (non exhaustive) description of this locus by means of the notion of *Radius of convergence* as a function on the Berkovich space (following the lines of [BV07]), and describe its logarithmic properties on the maximal Skeleton. Thank to this principle we are able to provide criterions for the *infinitesimality and non degeneracy* of families of automorphisms (cf. Lemmas 5.13 and 5.3). Also σ -compatibility condition can be tested on the finite set of critical points (cf. Lemma 5.5). In Section 3.3 we introduce the notion of analytic cocycle, which are a characterization of solutions of differential equations. Their radius of convergence coincides hence with that studied in the fundamental work [BV07] and is then a continuous function on the whole Berkovich space $\mathcal{M}(X)$. The “global continuity” of the radius is not used in the proofs of the present paper, but only its log-concavity properties on the maximal Skeleton (cf. Section 2.7). In this sense the paper is self contained concerning the notions about the Radius.

Acknowledgments

A particular thank goes to D.Barsky for the very suggesting and helpful discussions we had. The author wish to express also his gratitude to Gilles Christol for the inspiring discussions, and his constant encouragements. A particular thank goes to the kindness of L.Berger who pointed out us the estimation [Col, Lemma 9.4] and the results of [Ber07]. With a particular attention we thank very much A.Marmora for current discussions and for pointing out us the result [Mar04, Prop.5.2]. We had also interesting and useful discussions with O.Brinon and A.Escassut.

DEFINITIONS AND NOTATION

1. Rings of functions

In this paper $(K, |\cdot|)$ will be an ultrametric complete valued field of characteristic 0 (e.g. $\mathbb{C}((T))$, \mathbb{Q}_p , \mathbb{C}_p , ...). It is understood that the valuation of every complete valued field extension Ω/K extends that of K . We denote by

$$\omega := \liminf_n |n!|^{1/n} = \begin{cases} 1 & \text{if } |\cdot| \text{ is trivial on } \mathbb{Z}; \\ |p|^{\frac{1}{p-1}} & \text{if } |\cdot| \text{ induces the } p\text{-adic absolute value on } \mathbb{Z}. \end{cases} \quad (1.1)$$

If $|\cdot|$ is a seminorm on a ring A , we extend it to a seminorm on $M_{n \times n}(A) = M_n(A)$, by setting $|(a_{i,j})_{i,j}| := \max_{i,j} |a_{i,j}|$.

1.1 Analytic functions.

Let $\mathbb{R}_{\geq 0} := \{r \in \mathbb{R} \mid r \geq 0\}$, and let $\delta_1 := T \frac{d}{dT}$. We denote by

$$\mathcal{A}_K(c, R) := \left\{ \sum_{n \geq 0} a_n (T - c)^n \mid a_n \in K, \liminf_n |a_n|^{-1/n} \geq R \right\} \quad (1.2)$$

the ring of analytic functions on the disk $D^-(c, R)$, $c \in K$. Its topology is the finest one making continuous the family of norms $|\sum a_i (T - c)^i|_{c, \rho} := \sup |a_i| \rho^i$, for all $\rho < R$. $\mathcal{A}_K(c, R)$ is complete for this topology.

Let $\emptyset \neq I \subseteq \mathbb{R}_{\geq 0}$ be some interval. We denote the annulus relative to $c \in K$ and to I by

$$\mathcal{C}_c(I) := \{x \mid |x - c| \in I\}. \quad (1.3)$$

We denote by

$$\mathcal{A}_K(c, I) := \left\{ \sum_{i \in \mathbb{Z}} a_i (T - c)^i \mid a_i \in K, \lim_{i \rightarrow \pm \infty} |a_i| \rho^i = 0, \forall \rho \in I \right\} \quad (1.4)$$

the ring of analytic functions on $\mathcal{C}_c(I)$. We set $|\sum_i a_i (T - c)^i|_{c, \rho} := \sup_i |a_i| \rho^i < +\infty$, for all $\rho \in I$. The ring $\mathcal{A}_K(I)$ is complete for the topology given by the family of norms $\{|\cdot|_{c, \rho}\}_{\rho \in I}$. We denote by $\mathcal{A}_K(I) := \mathcal{A}_K(0, I)$.

The Robba ring is then defined as

$$\mathcal{R}_K := \bigcup_{\varepsilon > 0} \mathcal{A}_K(I_\varepsilon), \quad (1.5)$$

where $I_\varepsilon :=]1 - \varepsilon, 1[$, and is complete with respect the limit Frechet topology.

1.2 Affinoids

DEFINITION 1.1. A bounded *sub- K -affinoid* of \mathbb{A}_K^1 is an analytic subset of \mathbb{A}_K^1 defined by

$$X := D^+(c_0, R_0) - \bigcup_{i=1}^n D^-(c_i, R_i), \quad (1.6)$$

where $c_0, c_1, \dots, c_n \in K$, $0 < R_1, \dots, R_n \leq R_0$, and $c_1, \dots, c_n \in D^+(c_0, R)$. In the sequel we will call it simply affinoid or K -affinoid for simplicity.

We denote by X the K -affinoid itself, and for all ultrametric valued K -algebras $(L, |\cdot|)$, we denote by $X(L)$ its L -rational points (e.g. if $X = D^+(0, 1)$ then $X(K) = \mathcal{O}_K$ and $X(L) = \mathcal{O}_L$ are the rings of integers of K and L respectively). Notice that $X(K)$ can be the empty set.

We will write $X \otimes L$ the L -affinoid obtained by X by scalar extension. Identifying X with the functor $L \mapsto X(L)$, the $X \otimes L$ is the restriction of that functor to the extensions of L . Indeed, for all L'/L , one has $X(L') = (X \otimes L)(L')$. By abuse of notation, we will write sometimes X instead of $X \otimes L$.

Unless specified mention, in this paper all affinoids will be K -rationals, we will often omit the expression “ K -rational” in the sequel.

Let X be an affinoid, let $H_K^{\text{rat}}(X)$ be the ring of rational fractions $f(T)/g(T)$ in $K(T)$, without poles in $X(K^{\text{alg}})$, and let $\|\cdot\|_X$ be the norm on $H_K^{\text{rat}}(X)$ given by

$$\|f(T)\|_X := \sup_{\Omega/K} \left(\sup_{x \in X(\Omega)} |f(x)| \right), \quad (1.7)$$

where $(\Omega, |\cdot|)$ runs in the family of complete valued fields extension of $(K, |\cdot|)$ (it is understood that the absolute value of Ω extends that of K). Notice that $\|f\|_X = \sup_{x \in X(K^{\text{alg}})} |f(x)|$, for all $f \in H_K^{\text{rat}}(X)$. We denote the completion of $(H_K^{\text{rat}}(X), \|\cdot\|_X)$ by

$$\mathcal{H}_K(X). \quad (1.8)$$

If $\rho_1, \rho_2 \in |K^{\text{alg}}|$, and if $X = D^+(0, \rho_2) - D^-(0, \rho_1)$, then $\mathcal{H}_K(X) = \mathcal{A}_K(I)$, whit $I = [\rho_1, \rho_2]$.

Let $\varepsilon > 0$. If $X = D^+(c_0, R_0) - \bigcup_{i=1}^n D^-(c_i, R_i)$, we set

$$X_\varepsilon := D^+(c_0, R_0 + \varepsilon) - \bigcup_{i=1}^n D^-(c_i, R_i - \varepsilon). \quad (1.9)$$

We set then

$$\mathcal{H}_K^\dagger(X) := \bigcup_{\varepsilon > 0} \mathcal{H}_K(X_\varepsilon). \quad (1.10)$$

The ring $\mathcal{H}_K^\dagger(X)$ is complete with respect to the limit topology. If $X_1 := \{x \mid |x| = 1\}$ we set

$$\mathcal{H}_K := \mathcal{H}_K(X_1), \quad \mathcal{H}_K^\dagger := \mathcal{H}_K^\dagger(X_1). \quad (1.11)$$

2. Some properties of the Berkovich space of a bounded sub- K -affinoid of \mathbb{A}_K^1

DEFINITION 2.1 ([Ber90]). Let X be an affinoid. A *bounded multiplicative semi-norm* on $\mathcal{H}_K(X)$ is a function $|\cdot|_* : \mathcal{H}_K(X) \rightarrow \mathbb{R}_{\geq 0}$, such that $|0|_* = 0$, $|1|_* = 1$, $|f - g|_* \leq \max(|f|_*, |g|_*)$, $|fg|_* = |f|_* |g|_*$, and $|\cdot|_* \leq C \|\cdot\|_X$, for some constant $C > 0$. The *Berkovich space* $\mathcal{M}(\mathcal{H}_K(X)) = \mathcal{M}(X)$ is defined as the set of all bounded multiplicative semi-norms together with the finest topology making continuous the family of maps $\{|f| : \mathcal{M}(X) \rightarrow \mathbb{R}_{\geq 0}, |f|(|\cdot|_*) := |f|_*, |\cdot|_* \in \mathcal{M}(X)\}_{f \in \mathcal{H}_K(X)}$.

REMARK 2.2. Notice that C is always equal to 1, indeed since $|f^n|_* \leq C \|f^n\|_X$, then $|f|_* \leq C^{1/n} \|f\|_X$ for all $n \geq 0$. Moreover the restriction of $|\cdot|_*$ to K coincides with the absolute value $|\cdot|$ of K . Indeed since $\|a\|_X = |a|$, for all $a \in K$, the conditions $|a|_* \leq |a|$ and $|a^{-1}|_* \leq |a^{-1}|$, imply $|a|_* = |a|$.

2.1 Dwork generic points

Let $(\Omega, |\cdot|)/(K, |\cdot|)$ be a complete extension of valued field. Every point $c \in X(\Omega)$ produces the bounded multiplicative semi-norm $|\cdot|_c : f \mapsto |f(c)|_\Omega$ on $\mathcal{H}_K(X)$. This defines a continuous map

$$i_\Omega : X(\Omega) \rightarrow \mathcal{M}(X), \quad (2.1)$$

where $X(\Omega)$ has the topology induced by the absolute value of Ω (cf. [Ber90]). For all $|\cdot|_* \in \mathcal{M}(X)$, there exists a field extension $(\Omega, |\cdot|)/(K, |\cdot|)$ sufficiently large to have $|\cdot|_* = i_\Omega(t)$, for some $t \in X(\Omega)$. Such a point will be called a *Dwork generic point* attached to $|\cdot|_*$, of course t is not uniquely

determined by $|\cdot|_*$. The existence of the pair (Ω, t) is proved as follows. By Ω we can consider the completion of the fraction field of $\mathcal{H}_K(X)/\text{Ker}(|\cdot|_*)$ with respect to the norm induced by $|\cdot|_*$, then t is the image of $T \in \mathcal{H}_K(X)$. The composite map $K \rightarrow \mathcal{H}_K(X) \rightarrow \Omega$ is an isometry by Remark 2.2. Notice that the image of T in $\Omega = \widehat{\text{Frac}(\mathcal{H}_K(X)/\text{Ker}(|\cdot|_*))}$ is in some sense a canonical Dwork generic point, as far as the choice of T as variable is canonical.

2.2 Norms of the type $|\cdot|_{c,R}$.

Let $I \subseteq \mathbb{R}_{\geq 0}$ be an arbitrary interval. We recall that by definition a function $N : I \rightarrow \mathbb{R}_{\geq 0}$ has a given property logarithmically if its log-function $\tilde{N}(r) := \log(N(\exp(r))) : \log(I) \rightarrow [-\infty, +\infty]$ has that property.

DEFINITION 2.3. Let $\rho \geq 0$, and let c be an arbitrary point belonging to a complete valued extension Ω/K .² We define a multiplicative seminorm of the field of rational fractions $K(T)$ as follows. For a polynomial $P(T) \in K[T]$ we consider its developpement $P(T) = \sum_{i=0}^n a_i(T-c)^i$, $a_i \in \Omega$, at c , and we set

$$|P(T)|_{c,\rho} := \max_{i=0,\dots,n} |a_i| \rho^i. \quad (2.2)$$

This definition extends to a multiplicative seminorm of $K(T)$ and hence of $H_K^{\text{rat}}(X)$ in an evident way.

Let $X := D^+(c_0, R_0) - \cup_{i=1,\dots,n} D^-(c_i, R_i)$ be an affinoid. The seminorm $|\cdot|_{c,\rho}$ extends to a bounded multiplicative seminorm of $\mathcal{H}_K(X)$ if and only if three conditions are fulfilled:

- i) $c \in D^+(c_0, R_0)$,
- ii) $\rho \leq R_0$,
- iii) $\rho \geq R_i$ whenever $c \in D^-(c_i, R_i)$, for some $i = 1, \dots, n$.

REMARK 2.4 (Independence on c). We notice that if $|c - c'| \leq \rho$, then

$$|\cdot|_{c,\rho} = |\cdot|_{c',\rho}. \quad (2.3)$$

Indeed for all $a \in K^{\text{alg}}$ one has $|T - a|_{c,\rho} = |T - c + c - a|_{c,\rho} = \max(|c - a|, \rho) = \max(|c' - a|, \rho) = |T - a|_{c',\rho}$, hence, by multiplicativity, the equality hold for all polynomials (and hence for all rational fractions) with coefficients in K^{alg} . In particular it holds for every element of $K(T)$ and $H_K^{\text{rat}}(X)$, and hence of $\mathcal{H}_K(X)$ by density.

2.2.1 *Logarithmic properties.* Let $I \subset \mathbb{R}_{\geq 0}$ be an interval. Assume that the conditions i), ii), iii) after Definition 2.3 are fulfilled for all $\rho \in I$. Then the function

$$\rho \mapsto |\cdot|_{c,\rho} : I \rightarrow \mathcal{M}(X), \quad (2.4)$$

is *continuous*. More precisely, for all $f \in \mathcal{H}_K(X)$, the function $I \rightarrow \mathbb{R}_{\geq 0}$ sending ρ into $|f|_{c,\rho}$ is *continuous* and *piecewise log-affine* (i.e. piecewise of the form $C \cdot \rho^n$, for $C \in \mathbb{R}_{\geq 0}$, $n \in \mathbb{Z}$).

We examine now the case in which³

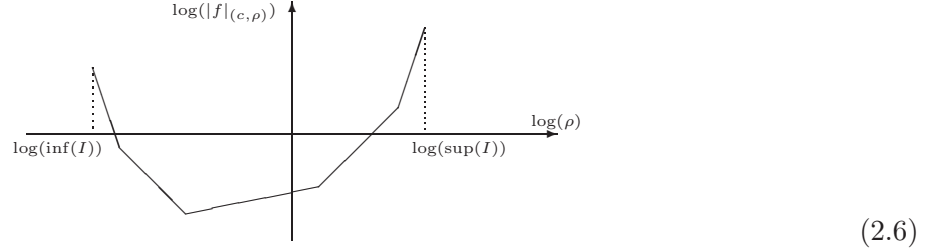
$$\mathcal{C}_c(I) \subset X, \quad (2.5)$$

where $\mathcal{C}_c(I) := \{ |x - c| \in I \}$ is the annulus centered at $c \in \Omega$ corresponding to the interval $I \subseteq \mathbb{R}_{\geq 0}$. In this case the function $\rho \mapsto |f|_{c,\rho}$ is moreover *log-convex*, and more precisely its log-function $|f|^\sim$ has a break at $\rho \in I$ if and only if f has a zero $\xi \in X(K^{\text{alg}})$ such that $|\xi - c| = \rho$. Moreover the

²Notice that c do not necessarily belongs to $X(\Omega)$.

³Notice that c does not necessarily belong to $X(\Omega)$, and that for $I = [0, R[$ we find $\mathcal{C}_c(I) = D(c, R)$.

multiplicity of the zero ξ equals the difference between the slope of $|f|^\sim$ at $\log(\rho)^+$ and the slope at $\log(\rho)^-$:



(2.6)

Finally if $f = \sum_{i \in \mathbb{Z}} a_i (T - c)^i$ is the developpement of f over the annulus $\mathcal{C}_c(I) \subseteq X$, then, for all $\rho \in I$, the semi-norm $|f|_{c, \rho}$ admits the following two expressions:

$$|f|_{c, \rho} = \sup_{i \in \mathbb{Z}} |a_i| \rho^i = \sup_{L/K} \sup_{|x-c|=\rho, x \in L} |f(x)|, \quad (2.7)$$

where L/K runs in the family of complete valued field extensions of K .

One proves these properties by reduction to the case of rational fractions in $H_K^{\text{rat}}(X)$ because it is dense in $\mathcal{H}_K(X)$. Then one reduces to the case K^{alg} because of the inclusion $\mathcal{H}_K(X) \subseteq \mathcal{H}_{K^{\text{alg}}}(X)$. Moreover over K^{alg} we are reduced to check these properties for polynomials of the type $(T - a)$, $a \in K^{\text{alg}}$, because of the multiplicativity (see [CR94, Chapter 2] for more details).

REMARK 2.5. The function $\rho \mapsto |f|_{c, \rho}$ is not globally logarithmically convex, but only on the intervals I such that $\mathcal{C}_c(I) \subset X$. In particular, if $c \in X(\Omega)$, then $\rho \mapsto |f|_{c, \rho}$ is log-convex on $I = [0, \rho_{c, X}[$ (cf. (2.9) below), this implies

$$|f(c)|_\Omega \leq |f|_{c, \rho}, \quad \text{for all } \rho \in [0, \rho_{c, X}[. \quad (2.8)$$

Observe moreover that the equality (2.7) is false if X has a hole contained in $\{|x - c| = \rho\}$.

2.3 The number $\rho_{y, X}$.

For all valued field Ω/K , and all $y \in X(\Omega)$ we set

$$\rho_{y, X} := \sup \{ R \mid D^-(y, R) \subset X \}, \quad (2.9)$$

where the inclusion of $D^-(y, R)$ in X is an inclusion of analytic spaces. In other words $\rho_{y, X} := \min_{\Omega'/K} \sup \{ R \mid D_{\Omega'}^-(y, R) \subset X(\Omega') \}$, where Ω' runs in the family of the complete valued fields extensions of K containing y . If $X := D^+(c_0, R_0) - \cup_{i=1, \dots, n} D^-(c_i, R_i)$, one sees that

$$\rho_{y, X} = \min(R_0, |y - c_1|, \dots, |y - c_n|), \quad (2.10)$$

and that, if t_{c_i, R_i} is a Dwork generic point for $|\cdot|_{c_i, R_i}$, then

$$r_X = \min_{y \in X(\Omega), \Omega/K} \rho_{y, X} = \min(R_0, R_1, \dots, R_n) = \min_{i=0, \dots, n} \rho_{t_{c_i, R_i}, X}. \quad (2.11)$$

Hence, if $R \leq r_X$, then for all valued field extension Ω/K , and all $y \in X(\Omega)$, the disc $D^-(y, R)$ is contained in $X \otimes \Omega$. More generally we can define the number $\rho_{|\cdot|_*, X}$ for every Berkovich point $|\cdot|_* \in \mathcal{M}(X)$. If $t_* \in X(\Omega)$ is a Dwork generic point for $|\cdot|_*$, then we set

$$\rho_{|\cdot|_*, X} := \rho_{t_*, X}. \quad (2.12)$$

LEMMA 2.6. *The definition of $\rho_{|\cdot|_*, X}$ does not depend on the choice of t_* . Moreover we have the following implication*

$$|\cdot|_* \leq |\cdot|_{**} \implies \rho_{|\cdot|_*, X} = \rho_{|\cdot|_{**}, X}. \quad (2.13)$$

Proof. By definition one has

$$\rho_{|\cdot|_*, X} = \min_{i=1, \dots, n} (R_0, |t_* - c_i|) = \min_{i=1, \dots, n} (R_0, |T - c_i|_*) \quad (2.14)$$

which does not depend on the choice of t_* . This expression of $\rho_{|\cdot|_*, X}$ provides the inequality $\rho_{|\cdot|_*, X} \leq \rho_{|\cdot|_{**}, X}$. To prove the converse inequality we now find another expression of $\rho_{|\cdot|_*, X}$. Let $\{b_s\}_{s \geq 0}$ be a sequence in K satisfying $\lim_s |b_s| R_0^s = 0$, and $\lim_s |b_s| R^s = +\infty$, for all $R > R_0$. The power series $f_0(T) = \sum_{s \geq 0} b_s (T - c_0)^s$ converges exactly on $D^+(c_0, R_0)$, and belongs hence to $\mathcal{H}_K(X)$. The function $f_X(T) := f_0(T) + \sum_{i=1}^n (T - c_i)^{-1} \in \mathcal{H}_K(X)$ has the property that its Taylor expansion at a point $y \in X(\Omega)$ (Ω/K being as usual an arbitrary complete valued field extension) converges with radius $\rho_{y, X}$. We can write $\rho_{y, X} = \text{Rad}(f_X, y)$, where $\text{Rad}(f_X, y) = \liminf_n (|f_X^{(n)}(y)|/|n!|)^{-1/n}$ is the radius of convergence of the Taylor expansion $f_X = \sum_{n \geq 0} f_X^{(n)}(y) (T - y)^n / n!$ of f_X at $y \in \Omega$. Since $|f_X^{(n)}(t_*)| = |f_X^{(n)}|_*$, then we can express $\rho_{|\cdot|_*, X}$ as

$$\rho_{|\cdot|_*, X} := \liminf_n (|f_X^{(n)}|_*/|n!|)^{-1/n}, \quad (2.15)$$

which proves one time more that it does not depend on t_* . The inequality $\rho_{|\cdot|_*, X} \geq \rho_{|\cdot|_{**}, X}$ follows then by this expression. \square

2.4 Shilov boundary

A *Shilov boundary* of X is a set of points $\mathcal{S} := \{|\cdot|_i\}_{i \in I} \subset \mathcal{M}(X)$ satisfying

$$\|f\|_X = \sup_{|\cdot|_i \in \mathcal{S}} |f|_i, \quad \text{for all } f \in \mathcal{H}_K(X). \quad (2.16)$$

Every affinoid $X = D^+(c_0, R_0) - \cup_{i=1, \dots, n} D^-(c_i, R_i)$ admits a *finite* Shilov boundary given by

$$\mathcal{S}_X = \{|\cdot|_{c_i, R_i}\}_{i=0, \dots, n}. \quad (2.17)$$

If t_{c_i, R_i} is a Dwork generic point associated to $|\cdot|_{c_i, R_i}$ one has then

$$\|f\|_X = \max_{i=0, \dots, n} |f|_{c_i, R_i} = \max_{i=0, \dots, n} |f(t_{c_i, R_i})|, \quad \text{for all } f \in \mathcal{H}_K(X). \quad (2.18)$$

LEMMA 2.7. *Let $X = D^+(c_0, R_0) - \cup_{i=1, \dots, n} D^-(c_i, R_i)$ be an affinoid. Let*

$$r_X := \min(R_0, \dots, R_n). \quad (2.19)$$

Then $\|\frac{d}{dT} f(T)\|_X \leq r_X^{-1} \|f(T)\|_X$. \square

2.5 Maximal Skeleton and critical points of the Berkovich space of a bounded sub- K -affinoid of \mathbb{A}_K^1

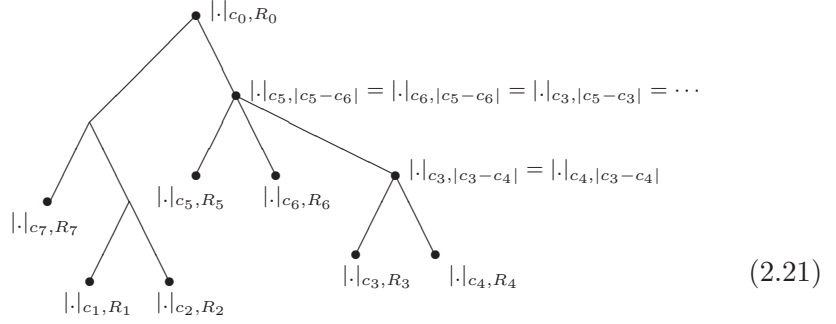
Let $X = D^+(c_0, R_0) - \cup_{i=1, \dots, n} D^-(c_i, R_i)$ be an affinoid.

DEFINITION 2.8. We call the *maximal skeleton* of X the following subset of $\mathcal{M}(X)$

$$\mathcal{S}_X := \bigcup_{i=1, \dots, n} \{|\cdot|_{c_i, \rho} \mid \rho \in [R_i, R_0]\}. \quad (2.20)$$

It is understood that if $X = D^+(c_0, R_0)$ has no holes, then the above family is reduced to the Shilov boundary $\mathcal{S}_X := \{|\cdot|_{c_0, R_0}\}$.

The union on the right hand side of (2.20) is not disjoint. Indeed, by Remark 2.4, for all $i, j = 1, \dots, n$ one has $|\cdot|_{c_i, \rho} = |\cdot|_{c_j, \rho}$, for all $\rho \geq |c_i - c_j|$.



(2.21)

The maximal Skeleton of an annulus $\mathcal{C}_c(I) := \{x \mid |x - c| \in I\}$ is given by

$$\mathcal{S}_{\mathcal{C}_c(I)} := \{|\cdot|_{c, \rho} \}_{\rho \in I}. \quad (2.22)$$

DEFINITION 2.9. We will call *critical* points, the points belonging to the family

$$\mathcal{P}_X := \{|\cdot|_{c_i, R_i}\}_{i=0, \dots, n} \cup \{|\cdot|_{c_i, |c_i - c_j|}\}_{i, j=1, \dots, n}. \quad (2.23)$$

Notice that the Shilov boundary is composed by critical points. One has the inclusions:

$$\mathcal{S}_X \subseteq \mathcal{P}_X \subseteq \mathcal{S}_X \subseteq \mathcal{M}(X), \quad (2.24)$$

where $\mathcal{S}_X = \{|\cdot|_{c_i, R_i}\}_{i=0, \dots, n}$ is the Shilov boundary of X , and \mathcal{P}_X is the set of critical points just defined, and $\mathcal{S}_X = \cup_{i=1}^n \{|\cdot|_{c_i, \rho}\}_{\rho \in [R_i, R_0]}$ is the maximal Skeleton.

REMARK 2.10. The interest of the maximal Skeleton \mathcal{S}_X and of the set of critical points \mathcal{P}_X lies in the fact that many properties of continuous functions on \mathcal{M} are verified in all point if and only if they are verified in the maximal Skeleton or in the finite set of critical points.

PROPOSITION 2.11. *The maximal Skeleton \mathcal{S}_X coincides with the set of maximal points of $\mathcal{M}(X)$.*⁴

Proof. We firstly prove that for all $|\cdot|_* \in \mathcal{M}(X)$ there exists a point $|\cdot|_{**} \in \mathcal{S}_X$ such that $|\cdot|_* \leq |\cdot|_{**}$. Let $t \in X(\Omega)$ be a Dwork generic point for $|\cdot|_*$. By Section 2.2.1 for every function $f \in \mathcal{H}_K(X)$ one has $|f|_* = |f(t)| \leq |f|_{t, \rho_{t, X}}$. If $X = D^+(c_0, R_0)$, the assertion follows from the fact that $|f|_{t, \rho_{t, X}} = |f|_{t, R_0} = |f|_{c_0, R_0}$, for all $f \in \mathcal{H}_K(X)$. Assume now that X has at least a hole. If $\rho_{t, X} = R_0$, then $|c_i - t| = R_0$ for all $i = 1, \dots, n$. Then we can write $|f|_* = |f(t)| = |f|_{t, 0} \leq |f|_{t, \rho_{t, X}} = |f|_{t, R_0} = |f|_{c_i, R_0}$, for all $f \in \mathcal{H}_K(X)$. Assume now that $\rho_{t, X} < R_0$. By definition one has $\rho_{t, X} = |t - c_i| \geq R_i$, for some $i = 1, \dots, n$, so that $|f|_* = |f(t)| \leq |f|_{t, \rho_{t, X}} = |f|_{c_i, \rho_{t, X}}$, for all $f \in \mathcal{H}_K(X)$. This proves that \mathcal{S}_X contains the set of maximal points of $\mathcal{M}(X)$. It hence sufficient to prove that the inequality $|\cdot|_* \leq |\cdot|_{**}$ implies $|\cdot|_* = |\cdot|_{**}$ for every couple of elements $|\cdot|_*, |\cdot|_{**} \in \mathcal{S}$. Assume that $|\cdot|_{c_i, \rho_i} \leq |\cdot|_{c_j, \rho_j}$. To prove that $|\cdot|_{c_i, \rho_i} = |\cdot|_{c_j, \rho_j}$ it is enough to prove that $\rho_i = \rho_j$, and that $|c_i - c_j| \leq \min(\rho_i, \rho_j)$ (cf. Remark 2.4). From $\max(\rho_i, |c_i - c_j|) = |(T - c_i) + (c_i - c_j)|_{c_i, \rho_i} = |T - c_j|_{c_i, \rho_i} \leq |T - c_j|_{c_j, \rho_j} = \rho_j$ we deduce $\rho_i \leq \rho_j$ and $|c_i - c_j| \leq \rho_j$. From $\rho_i^{-1} = |(T - c_i)^{-1}|_{c_i, \rho_i} \leq |(T - c_i)^{-1}|_{c_j, \rho_j} = |(T - c_j) + (c_j - c_i)|_{c_j, \rho_j}^{-1} = \max(\rho_j, |c_i - c_j|)^{-1}$ we deduce $\rho_i \geq \rho_j$ and $\rho_i \geq |c_i - c_j|$. Hence $\rho_i = \rho_j$ and $|c_i - c_j| \leq \min(\rho_i, \rho_j)$ as required. \square

It will be convenient to express explicitly the value of $\rho_{|\cdot|_*, X}$ for all $|\cdot|_*$ of the maximal Skeleton \mathcal{S}_X of X .

PROPOSITION 2.12. *Let $X := D^+(c_0, R_0) - \cup_{i=1, \dots, n} D^-(c_i, R_i)$. Let $R_i \leq \rho \leq R_0$. Then one has*

$$\rho_{|\cdot|_{c_i, \rho}, X} = \rho. \quad (2.25)$$

⁴Here we consider the natural partial order of semi-norms, namely $|\cdot|_* \leq |\cdot|_{**}$ if and only if $|f|_* \leq |f|_{**}$ for all $f \in \mathcal{H}_K(X)$.

Proof. By definition one has $\rho_{|\cdot|_{c_i, \rho}, X} = \min_{j=1, \dots, n} (R_0, |T - c_j|_{c_i, \rho})$. On the other hand $|T - c_j|_{c_i, \rho} = |(T - c_i) + (c_i - c_j)|_{c_i, \rho} = \max(\rho, |c_i - c_j|) \geq \rho$, hence the above minimum is greater than ρ . Since for $j = i$ we have $|T - c_j|_{c_i, \rho} = \rho$, the proposition is proved. \square

2.6 log-graphic of a function on the maximal Skeleton.

Let $X := D^+(c_0, R_0) - \cup_{i=1, \dots, n} D^-(c_i, R_i)$ be an affinoid. In this subsection we give a description of the logarithmic behavior of the log-graphic of a function $f \in \mathcal{H}_K(X)$ on the maximal Skeleton.

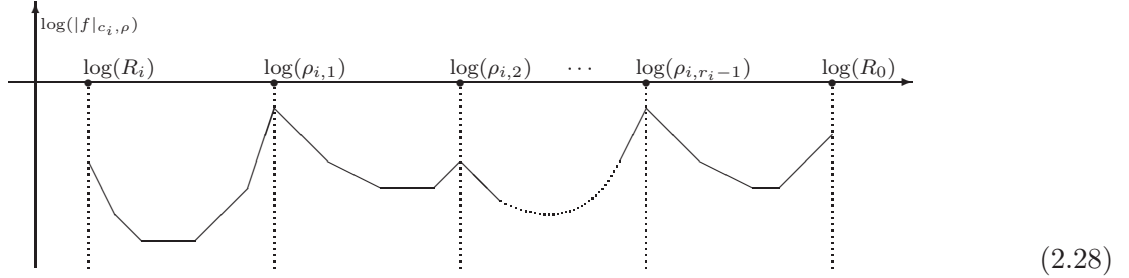
PROPOSITION 2.13. *Let $f \in \mathcal{H}_K(X)$. For a fixed index $i \in \{1, \dots, n\}$ consider the path $\{|\cdot|_{c_i, \rho}\}_{R_i \leq \rho \leq R_0}$ in $\mathcal{M}(X)$. Consider the following sub-set of critical points*

$$\mathcal{P}_{X,i} := \{R_i, R_0\} \cup \{|c_i - c_j|\}_{j=1, \dots, n, j \neq i}. \quad (2.26)$$

We now consider $\mathcal{P}_{X,i}$ as an ordered the set. We denote by

$$R_i = \rho_{i,0} < \rho_{i,1} < \rho_{i,2} < \dots < \rho_{i,r_i} = R_0 \quad (2.27)$$

the increasing sequence of elements of $\mathcal{P}_{X,i} = \{\rho_{i,0}, \dots, \rho_{i,r_i}\}$. Then the log-graphic of $\rho \mapsto |f|_{c_i, \rho}$ is continuous and log-convex in $[\rho_{i,s}, \rho_{i,s+1}]$, for all $0 \leq s \leq r_i - 1$.



Proof. This follows from section 2.2.1 since the annulus $\mathcal{C}_c([\rho_s, \rho_{s+1}])$ is included in X , and we have the inclusion $\mathcal{H}_K(X) \subset \mathcal{A}_K([\rho_s, \rho_{s+1}])$. \square

2.7 Convergent functions on neighborhoods of the diagonal

A typical example of neighborhoods of the diagonal is given by the *tubular* neighborhoods, that is those of the form

$$\mathcal{T}(X, R) := \{(x, y) \in X \times X \mid |x - y| < R\}. \quad (2.29)$$

For $R \leq r_X$, the isomorphism $\mathcal{T}(X, R) \xrightarrow{\sim} D^-(0, R) \times X$ defined by $(x, y) \mapsto (\delta, y)$, with $\delta = \delta(x, y) = x - y$, induces an isomorphism of the ring $\mathcal{A}_K(\mathcal{T}(X, R))$ of analytic functions on $\mathcal{T}(X, R)$ with the tensor product $\mathcal{A}_{\mathcal{H}_K(X)}(0, R) := \mathcal{A}_K(0, R) \hat{\otimes}_K \mathcal{H}_K(X)$. Every function of $\mathcal{A}_{\mathcal{H}_K(X)}(0, R)$ can be uniquely written as $\sum_{n \geq 0} f_n(y) \delta^n$, with $f_n \in \mathcal{H}_K(X)$, satisfying $\lim_n \|f_n\|_X \rho^n = 0$ for all $\rho < R$. Hence, by this isomorphism, one sees that the elements of $\mathcal{A}_K(\mathcal{T}(X, R))$ can be uniquely written as (cf. [Ber90, before Prop.2.1.2])

$$f(x, y) = \sum_{n \geq 0} f_n(y) (x - y)^n, \quad (2.30)$$

with $f_n \in \mathcal{H}_K(X)$, satisfying $\lim_{n \rightarrow \infty} \|f_n\|_X \cdot \rho^n = 0$, for all $\rho < R \leq r_X$.

The convergence locus of such a function is usually not reduced to $\mathcal{T}(X, R)$. For example, if $X' \subset X$ is a sub-affinoid, then the inequality $\|f_n\|_{X'} \leq \|f_n\|_X$ implies that the restriction of $f(x, y)$ to $X' \times X'$ may converges on $\mathcal{T}(X', R')$, with $R' \geq R$.

REMARK 2.14. G.Christol pointed out to us that the convergence locus depends also on the way in which the function $f(x, y)$ is written. For example if $X = D^+(0, 1)$, and if $f(x, y) = \sum_n (x - y)^n$,

then it converges for $|x - y| < 1$, $|x|, |y| \leq 1$. Its re-summation $f(x, y) = \sum_{i,j=0}^{\infty} \binom{i+j}{i} x^i y^j$, converges for $|x|, |y| < 1$ and we “loose convergence”.

It seems complicate to describe completely the nature of the convergence locus of $f(x, y)$ (cf. section 3.2), in the following we give a partial description.

DEFINITION 2.15. For all complete valued field extension Ω/K , and all point $y_0 \in X(\Omega)$ we consider $f(x, y_0) \in \mathcal{A}_K(y_0, R)$ as a function of the single variable x . We set

$$\text{Rad}(f(x, y), y_0) := \min(\text{Rad}(f(x, y_0)) , \rho_{y_0, X}) , \quad (2.31)$$

where

$$\text{Rad}(f(x, y_0)) = \liminf_{n \geq 0} |f_n(y_0)|_{\Omega}^{-1/n} \quad (2.32)$$

is the radius of convergence of $f(x, y_0)$ as function of x . We extend this definition to $\mathcal{M}(X)$ by

$$\text{Rad}(f(x, y), |\cdot|_*) := \text{Rad}(f(x, y), t_*) , \quad (2.33)$$

where t_* is a Dwork Generic point for $|\cdot|_*$. We call $\text{Rad}(f(x, y), |\cdot|_*)$ the *Radius of convergence of $f(x, y)$ at $|\cdot|_*$* . By equations (2.32) and (2.12) it is clear that this definition does not depend on the choice of t_* .

LEMMA 2.16. *The convergence locus of $f(x, y)$ contains the set*

$$\{ (x_0, y_0) \mid |x_0 - y_0| < \text{Rad}(f(x, y), y_0) \} . \quad (2.34)$$

LEMMA 2.17 (Transfer principle). *If $|\cdot|_* \leq |\cdot|_{**}$, then $\text{Rad}(f(x, y), |\cdot|_*) \geq \text{Rad}(f(x, y), |\cdot|_{**})$.*

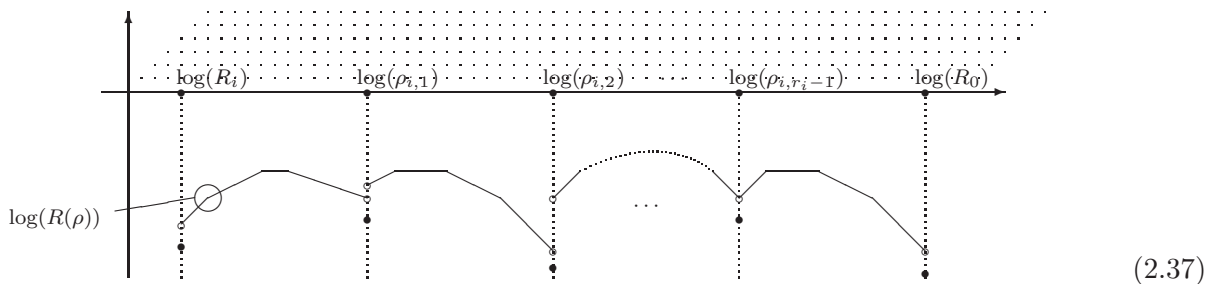
Proof. It follows from $\text{Rad}(f(x, y), |\cdot|_*) = \liminf_n |f_n|_*^{-1/n} \geq \liminf_n |f_n|_{**}^{-1/n} = \text{Rad}(f(x, y), |\cdot|_{**})$, and from Lemma 2.6. \square

PROPOSITION 2.18 (log-concavity of $\text{Rad}(f(x, y), -)$). *Let $X = D^+(c_0, R_0) - \cup_{i=1}^n D^-(c_i, R_i)$. With the notations of Proposition 2.13 the function $\rho \mapsto \text{Rad}(f(x, y), |\cdot|_{c_i, \rho})$ is log-concave in every interval $[\rho_{i,s}, \rho_{i,s+1}]$, and continuous in $] \rho_{i,s}, \rho_{i,s+1}[$, for all $s = 0, \dots, r_i - 1$. In particular*

$$\lim_{\rho \rightarrow \rho_{i,s}^{\pm}} \text{Rad}(f(x, y), |\cdot|_{c_i, \rho}) \geq \text{Rad}(f(x, y), |\cdot|_{c_i, \rho_{i,s}}) . \quad (2.35)$$

The following picture represents the log-graphic of the function $\rho \mapsto R(\rho)$ where

$$R(\rho) := \text{Rad}(f(x, y), |\cdot|_{c_i, \rho}) / \rho_{|\cdot|_{c_i, \rho}, X} = \text{Rad}(f(x, y), |\cdot|_{c_i, \rho}) / \rho .^5 \quad (2.36)$$



Proof. The lim inf of log-concave functions is log-concave (cf. Prop. 2.13). A log-concave function is continuous on open intervals. \square

⁵Notice that $\rho_{|\cdot|_{c_i, \rho}, X} = \rho$ by Proposition 2.12, and hence that the log-graphic of the function $\log(\rho) \mapsto R(\rho)$ is above the horizontal line $\log(\rho) \mapsto \log(\rho_{|\cdot|_{c_i, \rho}, X} / \rho) = 1$ by definition of the radius.

REMARK 2.19. The presence of $\rho_{y_0, X}$ in the definition is due to the fact that we want to have the property $\text{Rad}(g(x) \cdot f(x, y), y_0) \geq \text{Rad}(f(x, y), y_0)$, for all $g(x) \in \mathcal{H}_K(X)$. Indeed $g(x)$ converges on $D^-(y_0, \rho_{y_0, X})$, but may not converge outside X (cf. Corollary 3.10). This definition of Radius comes back to B.Dwork who introduced the idea of study the radius of convergence of solutions of differential equations at (today's called) ‘‘Dwork generic points’’ (cf. Section 2.1). Nevertheless the definition of Dwork involves the number $\tilde{\rho}_t := \max\{R \mid D_\Omega^-(t, R) \cap K = \emptyset\}$, $t \in X(\Omega)$, instead of $\rho_{t, X}$. In the most important cases, namely for the point of the maximal Skeleton, one has $\rho_{t, X} = \tilde{\rho}_t$ (cf. Prop. 2.12). The introduction of $\rho_{t, X}$ in the definition is due to [BV07], who proved the continuity of the function $|\cdot|_* \mapsto \text{Rad}(Y(x, y), |\cdot|_*)$, when $Y(x, y)$ is the Taylor solution of a differential equation with coefficients in $\mathcal{H}_K(X)$. The work of [BV07] generalizes to Berkovich spaces the preceding theorems of [CD94] about the continuity of this function on the maximal Skeleton of an annulus. Indeed on an annulus one has $\rho_{|\cdot|_\rho, \mathcal{C}(I)} = \tilde{\rho}_{t_\rho} = \rho$ (here t_ρ is a Dwork generic point attached to $|\cdot|_\rho := |\cdot|_{0, \rho}$ on the annulus $\mathcal{C}(I) = \{|x| \in I\}$).

2.7.1 *Global Radius of convergence.* We call *Global Radius of convergence of $f(x, y)$ with respect to X* the quantity

$$\text{Rad}(f(x, y), X) := \min(r_X, \min_{\Omega/K, y_0 \in X(\Omega)} \text{Rad}(f(x, y_0))) , \quad (2.38)$$

$$= \min_{\Omega/K, y_0 \in X(\Omega)} \text{Rad}(f(x, y), y_0) , \quad (2.39)$$

where Ω/K runs in the family of all complete valued field extensions of K . The second equality follows from equation (2.11). Clearly

$$\text{Rad}(f(x, y), X) = \sup\{R \mid R \leq r_X, \text{ and } f(x, y) \text{ converges on } \mathcal{T}(X, R)\} . \quad (2.40)$$

LEMMA 2.20. *Let $X = D^+(c_0, R_0) \cup_{i=1, \dots, n} D^-(c_i, R_i)$, $c_0, c_1, \dots, c_n \in K$, $R_0, R_1, \dots, R_n > 0$, be a K -affinoid. Let $f(x, y) = \sum_{s \geq 0} f_s(y)(x - y)^s$ be a K -rational analytic function converging on a tubular neighborhood of the diagonal of $X \times X$. Let $\mathcal{S}_X := \{|\cdot|_{c_0, R_0}, |\cdot|_{c_1, R_1}, \dots, |\cdot|_{c_n, R_n}\}$ be the Shilov boundary of X , and let $\{t_{c_0, R_0}, t_{c_1, R_1}, \dots, t_{c_n, R_n}\}$ be a family of corresponding Dwork generic points. Then*

$$\text{Rad}(f(x, y), X) = \min_{i=0, 1, \dots, n} (\text{Rad}(f(x, y), t_{c_i, R_i})) = \min(r_X, \liminf_s \|f_s\|_X^{-1/s}) . \quad (2.41)$$

Proof. The second equality follows by the relation $\|f_s\|_X = \max_{i=0, \dots, n} |f_s(t_{c_i, R_i})|$, since ‘‘lim inf’’ commutes with the ‘‘minimum over a finite family’’. Now we check the first equality. One sees that the inequality \leq holds because for $y_0 = t_{c_i, R_i}$, the term $\text{Rad}(f(x, y), t_{c_i, R_i})$ appears in the expression (2.39). The converse inequality follows then by the fact that $\min(r_X, \liminf_s \|f_s\|_X^{-1/s}) \leq \min(r_X, \min_{\Omega/K, y_0 \in X(\Omega)} \text{Rad}(f(x, y_0)))$. Indeed $\liminf_s \|f_s\|_X^{-1/s} = \liminf_s \inf_{\Omega/K, y_0 \in X(\Omega)} |f_s(y_0)|^{-1/s} \leq \inf_{\Omega/K, y_0 \in X(\Omega)} \liminf_s |f_s(y_0)|^{-1/s} = \inf_{\Omega/K, y_0 \in X(\Omega)} \text{Rad}(f(x, y_0))$. \square

FIRST PART: DEFORMATION FUNCTOR

3. Linear Differential Equations

Let (B, d) be a ring together with a fixed derivation $d : B \rightarrow B$. A *differential module over B* is a finite free B -module M , together with a connection on M . We recall that a connection on M is a linear map $\nabla^M : M \rightarrow M$ satisfying $\nabla(bm) = d(b) \cdot m + b \cdot \nabla(m)$, for all $b \in B$, $m \in M$. A morphism of differential modules $\alpha : (M, \nabla^M) \rightarrow (N, \nabla^N)$ is a B -linear morphism commuting with the actions of ∇ on M and N . The category of differential modules with coefficients in B will be indicated with

$$d - \text{Mod}(B) . \quad (3.1)$$

This is a \otimes -category in the sense of [DM]. The unit object is given by (B, d) . The internal \otimes and Hom are given by $(M \otimes_B N, \nabla^{M \otimes N})$ and $(\text{Hom}_B(M, N), \nabla^{\text{Hom}(M, N)})$, where $\nabla^{M \otimes N}$ and $\nabla^{\text{Hom}(M, N)}$ are defined by

$$\nabla^{M \otimes N}(m \otimes n) := \nabla^M(m) \otimes n + m \otimes \nabla^N(n), \quad m \in M, n \in N, \quad (3.2)$$

$$\nabla^{\text{Hom}(M, N)}(\phi)(m) := \nabla^N(\phi(m)) - \phi(\nabla^M(m)), \quad \phi \in \text{Hom}_B(M, N). \quad (3.3)$$

If $\mathbf{e} := \{e_1, \dots, e_n\} \subset M$ is a basis of M , the connection ∇^M is determined by its values on \mathbf{e} . If $\nabla^M(e_i) = \sum_{j=1}^n g_{i,j} e_j$ we call $G := (g_{i,j})_{i,j=1,\dots,n}$ the matrix of ∇^M in the basis \mathbf{e} .

3.0.2 Solutions, formal definition. Let B' be a B -algebra, together with a derivation d' , extending d . Let $M \in d\text{-Mod}(B)$. A solution of M with values in B' is a B -linear map $\alpha : M \rightarrow B'$ satisfying $d' \circ \alpha = \alpha \circ \nabla^M$. Let $(B')^{d=0} \subset B'$ be the sub-ring formed by elements b' such that $d'(b') = 0$. The solutions of M with values in B' forms a $(B')^{d=0}$ -module.

If G is the matrix of ∇^M in the basis $\mathbf{e} = \{e_1, \dots, e_n\}$, then the vector $\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in (B')^n$, defined by $y_i := \alpha(e_i)$, verifies $\begin{pmatrix} d'(y_1) \\ \vdots \\ d'(y_n) \end{pmatrix} = G \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$. We will say hence that M is defined in the basis \mathbf{e} by the (linear differential) equation

$$d(Y) = G \cdot Y. \quad (3.4)$$

LEMMA 3.1. Assume that there exists a differential ring $(B', d')/(B, d)$, such that i) the map $B \rightarrow B'$ is injective, ii) M is trivialized by B' (i.e. $M \otimes B'$ is trivial as B' -differential module), iii) $B^{d=0}$ is a field, and $(B')^{d=0} = B^{d=0}$. Then the $B^{d=0}$ -vector space of solutions of M , with values in B has $B^{d=0}$ -dimension smaller than or equal to $\text{rank}_B M$. \square

Proof. Let $n := \dim_B M$. Since $B \subset B'$ is injective, so does the natural map $\text{Hom}_B(M, B) \subset \text{Hom}_B(M, B')$. Hence the canonical linear map of $B^{d=0}$ -vector spaces $\text{Hom}_B^\nabla(M, B) \subset \text{Hom}_B^\nabla(M, B') = \text{Hom}_{B'}^\nabla(M \otimes B', B')$ is injective too. Hence we can assume $B = B'$. By ii) $M \otimes B'$ is trivial, i.e. isomorphic to $(B')^n$ as B' -differential module, so $\text{Hom}_B^\nabla((B')^n, B') = \text{Hom}_B^\nabla(B', B')^n$. By iii) one has $\text{Hom}_B^\nabla(B', B') = (B')^{d=0} = B^{d=0}$, this proves the lemma. \square

A fundamental matrix of solutions of M is a matrix $Y \in GL_n(B')$ satisfying Equation (3.4). Its columns define a basis of solution of M with values in B' .

3.1 Taylor Solutions

Consider now a $\mathcal{H}_K(X)$ -differential module M , and fix a basis $\mathbf{e} = \{e_1, \dots, e_n\}$ of M . The triplet $(M, \nabla^M, \mathbf{e})$ defines an equation

$$\frac{d}{dx}(Y) = G(x) \cdot Y, \quad G(x) \in M_n(\mathcal{H}_K(X)). \quad (3.5)$$

Let Ω/K be an extension of valued fields, and let $y \in X(\Omega)$. The Taylor formula express the fundamental basis of solution $Y(x, y)$ at y as

$$Y(x, y) = \sum_{n \geq 0} G_n(y) \frac{(x - y)^n}{n!}, \quad (3.6)$$

where $G_n(x)$ is defined by the relation $(d/dx)^n(Y) = G_n Y$. Inductively one has $G_{n+1} = d/dx(G_n) + G_n \cdot G$, $G_0 = \text{Id}$, $G_1 = G$. One proves easily by induction that $\|G_n(x)\|_X \leq \max(\|G_n\|_X, r_X)^n$, and $|n!| \geq \omega^n$, so the entries of $Y(x, y)$ are analytic functions on $\mathcal{T}(X, R)$ with $R = \omega \cdot \max(\|G_n\|_X, r_X)^{-1}$. Hence the properties of section 2.7 holds for $Y(x, y)$.

For all $y_0 \in X(\Omega)$, the matrix solution $Y(x, y_0)$ can be understood as a solution (in the formal sense) with values in $\mathcal{A}_K(y_0, R)$, for all $R \leq \text{Rad}(Y(x, y_0), y_0)$.

LEMMA 3.2 ([Pul07],[CM02]). Let $Y(x, y)$ be the Taylor Solution of the equation (3.5). For all complete valued field Ω/K , and all $y_0 \in X(\Omega)$, denote by $R_{y_0} := \text{Rad}(Y(x, y), y_0)$. Then:

- i) For all $z_1, z_2 \in X(\Omega)$, one has $z_1 \in D^-(z_2, R_{z_2})$ if and only if $z_2 \in D^-(z_1, R_{z_1})$. Equivalently if $|z_1 - z_2| < R_{z_2}$, then $R_{z_1} = R_{z_2}$.
- ii) For all $x \in X(\Omega)$, one has $Y(x, x) = \text{Id}$.
- iii) For all $c, x, y, z \in X(\Omega)$, such that $x, y, z \in D^-(c, R_c)$, one has $Y(x, y)Y(y, z) = Y(x, z)$.
- iv) For all $x, y \in X(\Omega)$ such that $x \in D^-(y, R_y)$ one has $Y(x, y)^{-1} = Y(y, x)$.
- v) One has $\frac{d}{dx}Y(x, y) = G(x) \cdot Y(x, y)$ (cf. equation (3.9)).

3.2 Convergence locus of a Taylor solution: uniform neighborhoods of the diagonal

If $Y(x, y)$ is the Taylor solution of a differential equation on X , then by Lemma 2.16 the following subset of $X \times X$ is contained in convergence locus of $Y(x, y)$:

$$\mathcal{U} := \{ (x_0, y_0) \mid |x_0 - y_0| < \text{Rad}(Y(x, y), y_0) \}. \quad (3.7)$$

We notice that the function $|\cdot|_* \mapsto \text{Rad}(Y(x, y), |\cdot|_*)$ is continuous on the whole $\mathcal{M}(X)$ by [BV07]. In particular the function $\rho \mapsto \text{Rad}(Y(x, y), |\cdot|_{c_i, \rho})$ of Proposition 2.18 is continuous on the whole interval $[R_i, R_0]$ in this case. There are many properties that \mathcal{U} satisfies, and its description can certainly be made more precise. Indeed for example it is known that the function $\rho \mapsto \text{Rad}(Y(x, y), |\cdot|_{c_i, \rho})$ satisfies important properties, as for example the fact that its slopes are rational with denominator bounded by $r!$, where r is the rank of $Y(x, y)$ (cf. [Pon00]). A precise description of these sets seems however difficult and lies outside our scopes. We give here a simple list of properties of these sets

DEFINITION 3.3. We call *uniform neighborhood of the diagonal* a (functorial) subset \mathcal{U} of $X \times X$ which satisfies:

- i) \mathcal{U} is defined as (cf. Lemma 2.16)

$$\mathcal{U} := \{ (x, y) \mid |x - y| < R_{\mathcal{U}}(|\cdot|_y) \}, \quad (3.8)$$

where $R_{\mathcal{U}} : \mathcal{M}(X) \rightarrow \mathbb{R}_{\geq 0}$ is a continuous function;

- ii) $R_{\mathcal{U}}$ satisfies $R_{\mathcal{U}}(|\cdot|_*) \leq \rho_{|\cdot|_*, X}$ for all $|\cdot|_* \in \mathcal{M}(X)$;
- iii) For all complete valued extension Ω/K , and all $(x, y) \in X(\Omega)$ satisfying $|x - y| < R_{\mathcal{U}}(|\cdot|_y)$, then $R_{\mathcal{U}}(|\cdot|_y) = R_{\mathcal{U}}(|\cdot|_x)$;
- iv) For all complete valued extension Ω/K , and all $(x, y) \in \mathcal{U}(\Omega)$, then $(y, x) \in \mathcal{U}(\Omega)$;⁶
- v) \mathcal{U} contains a tubular neighborhood $\mathcal{T}(X, R)$, for some $R > 0$.
- vi) For all complete valued extension Ω/K , and all $(x, y), (y, z) \in \mathcal{U}(\Omega)$, one has $(x, z) \in \mathcal{U}(\Omega)$.

3.2.1 *Uniform Neighborhoods of the diagonal, the case of an annulus.* Let $\mathcal{C}(I)$ be an annulus. If I is compact, then $\mathcal{C}(I)$ is an affinoid and the above definition applies. Assume then I non compact. A neighborhood of the diagonal of $\mathcal{C}(I) \times \mathcal{C}(I)$ is called uniform if, for every compact $J \subset I$, its restriction to $\mathcal{C}(J) \times \mathcal{C}(J)$ is uniform in the sense of the previous definition. Notice that a uniform neighborhood of the diagonal of $\mathcal{C}(I)$ does not necessarily contain a tabular neighborhood.

⁶The notation $\mathcal{U}(\Omega)$ is somehow abusive here because we have not proved that \mathcal{U} is an analytic sub-space of $X \times X$. By $\mathcal{U}(\Omega)$ we mean the set of $(x, y) \in X(\Omega) \times X(\Omega)$ satisfying the condition expressed by (3.8).

3.3 Analytic cocycles

DEFINITION 3.4 (Analytic cocycle). Let X be an affinoid. We call K -rational analytic cocycle a square matrix $Y(x, y)$ satisfying:

- i) The entries of $Y(x, y)$ are analytic functions on a tubular neighborhood of the diagonal $\mathcal{T}(X, R)$ (cf. Section 2.7) for some unspecified $0 < R \leq r_X$;
- ii) For all complete valued field extension Ω/K and all $x \in X(\Omega)$ one has $Y(x, x) = \text{Id}$;
- iii) For all complete valued field extension Ω/K and all $x, y, z \in X(\Omega)$ such that $|x - y|, |y - z|, |x - z| < R$, one has $Y(x, y) \cdot Y(y, z) = Y(x, z)$. In particular $Y(x, y)^{-1} = Y(y, x)$.

The following proposition provides a characterization of the Taylor solutions of differential equations with coefficients in $\mathcal{H}_K(X)$.

PROPOSITION 3.5 (Characterization of solutions of Differential equations). Let $Y(x, y)$ be a matrix whose entries are two variable analytic function on $\mathcal{T}(X, R)$, $0 < R \leq r_X$, satisfying $Y(x, x) = \text{Id}$. Then $Y(x, y)$ is an analytic cocycle if and only if it is the Taylor solution of a (unique) differential equation $\frac{d}{dx}Y(x, y) = G(x) \cdot Y(x, y)$, with $G(x) \in M_n(\mathcal{H}_K(X))$.

Proof. Let $h \in K$ be such that $|h| < R \leq r_X$. Then the matrix $A(h, x) := Y(x + h, x)$ belongs to $\mathcal{H}_K(X)$, and verifies $Y(x + h, y) = A(h, x)Y(x, y)$. Indeed if $Y(x, y) = \sum_{n \geq 0} H_n(y)(x - y)^n$ (cf. Section 2.7), then the series of functions $A(h, x) := \sum_{n \geq 0} H_n(x)h^n$ converges on X because $\lim_n \|H_n\|_X |h|^n = 0$ since $Y(x, y)$ converges on $\mathcal{T}(X, R)$, and $|h| < R$. One sees moreover that $A(0, x) := \text{Id}$, and that (by composition) $A(h, x)$ is analytic with respect to h . We can hence consider $G(x) := [d/dh(A(h, x))]_{|h=0} \in M_n(\mathcal{H}_K(X))$. By definition one has

$$\frac{d}{dx}(Y(x, y)) = \lim_{h \rightarrow 0} \frac{Y(x + h, y) - Y(x, y)}{h} = \lim_{h \rightarrow 0} \left(\frac{A(h, x) - \text{Id}}{h} \right) \cdot Y(x, y) = G(x) \cdot Y(x, y). \quad (3.9)$$

This proves that an analytic cocycle is solution of a differential equation. Reciprocally write $Y(x, y) = \text{Id} + \sum_{n \geq 1} H_n(y)(x - y)^n$ and $d/dx(Y(x, y)) = G(x)Y(x, y)$. For all complete valued field Ω/K , and all $y_0 \in X(\Omega)$, we specialize these equation at $y = y_0$. By the Taylor formula one finds $Y(x, y_0) = \sum_{n \geq 0} (d/dx)^n(Y(x, y_0))|_{x=y_0} (x - y_0)^n/n!$, hence $H_n(y_0) = (d/dx)^n(Y(x, y_0))|_{x=y_0}/n! = [G_n(x)Y(x, y_0)/n!]|_{x=y_0} = G_n(y_0)/n!$, where G_n are inductively defined by the relation $(d/dx)^n(Y) = G_n Y$, namely $G_0 = \text{Id}$, $G_1 = G$, $G_{n+1} = G'_n + G_n G$ (cf. Section 3.1). Hence $H_n(x) = G_n(x)/n!$ in $\mathcal{H}_K(X)$ (because they coincides as functions) and then $Y(x, y)$ equals the Taylor solution. \square

COROLLARY 3.6 (Rigidity and continuity of the Radius). Let $Y(x, y) = \sum_{n \geq 0} H_n(y)(x - y)^n$ be an analytic cocycle. Then

- i) One has $d/dx(Y(x, y)) = H_1(x)Y(x, y)$, i.e. the matrix $G(x)$ of Prop. 3.5 is equal to $H_1(x)$.
- ii) One has $H_0 = \text{Id}$, and the matrices $\{H_n(y)\}_{n \geq 2}$ are completely determined by $H_1(y)$ via the recursive relation $H_{n+1}(y) = (n + 1)^{-1}[d/dx(H_n(y)) + H_n(y)H_1(y)]$.
- iii) The radius of convergence $|\cdot|_* \mapsto \text{Rad}(Y(x, y), |\cdot|_*)$ is continuous on the whole $\mathcal{M}(X)$.
- iv) $Y(x, y)$ converges on the uniform neighborhood of the diagonal \mathcal{U} (cf. Definition 3.3) defined by the function $R_{\mathcal{U}} = \text{Rad}(Y(x, y), |\cdot|_*)$, and the rule iii) of Definition 3.4 holds for all $(x, y), (y, z), (x, z) \in \mathcal{U}(\Omega)$.

Proof. Point i) have been proved in the proof of Proposition 3.5. Point ii) follows by the fact that $Y(x, y)$ is forced to be the Taylor solution of the equation $Y' = H_1(x)Y$ (cf. Lemma 3.7, Section 3.1). The continuity of $\text{Rad}(Y(x, y), |\cdot|_*)$ have been proved in [BV07]. Point iv) follows from the fact that $Y(x, y)$ coincides with the Taylor solution of $Y' = GY$ (cf. Section 3.1). \square

LEMMA 3.7 (Uniqueness of the cocycle). *A (linear) differential equation $Y' = G \cdot Y$ has an unique analytic cocycle as solution: its Taylor solution.*

Proof. Assume that $Y_1' = GY_1$, and $Y_2' = GY_2$. Since $0 = \text{Id}' = (Y_1 Y_1^{-1})' = Y_1' Y_1^{-1} + Y_1 (Y_1^{-1})'$, then $d/dx(Y_1(x, y)^{-1}) = -Y_1(x, y)^{-1} G(x)$. Hence $d/dx(F(x, y)) = 0$, where $F(x, y) = Y_1(x, y)^{-1} Y_2(x, y)$. The function $F(x, y)$ does not depend on x , and for $x = y$ one has $F(y, y) = \text{Id}$, so $F(x, y) = \text{Id}$ for all $(x, y) \in \mathcal{T}(X, R)$, with R sufficiently small. So $Y_1(x, y) = Y_2(x, y)$. \square

LEMMA 3.8 (Rigidity). *Let Ω/K be a complete field extension and let $y_0 \in X(\Omega)$. If $Y_1(x, y)$, $Y_2(x, y)$ are two K -rational analytic cocycles such that $Y_1(x, y_0) = Y_2(x, y_0)$ as power series matrices in $GL_n(\mathcal{A}_\Omega(y_0, R))$, then $Y_1(x, y) = Y_2(x, y)$.*

Proof. Let $\frac{d}{dx} Y_s(x, y) = G^{[s]}(x) Y_s(x, y)$, $s = 1, 2$, be the differential equations satisfied by Y_1, Y_2 . Specializing at $y = y_0$ we find $\frac{d}{dx} Y_s(x, y_0) = G^{[s]}(x) Y_s(x, y_0)$, $s = 1, 2$. Since the morphism $\mathcal{H}_K(X) \rightarrow \mathcal{A}_\Omega(y_0, R)$ is injective, and since $Y_1(x, y_0) = Y_2(x, y_0)$, one has $G^{[1]}(x) = d/dx(Y_s(x, y_0)) Y_s(x, y_0)^{-1} = G^{[2]}(x)$ in $\mathcal{H}_K(X)$. By the point i) of Cor. 3.6, we $Y_1(x, y) = Y_2(x, y)$. \square

LEMMA 3.9 (Base change formulas). *Let $(M, \nabla^M, \mathbf{e})$ be a differential module, together with a fixed basis \mathbf{e} . Let $H(x) \in GL_n(\mathcal{H}_K(X))$ be a base change matrix. Assume that $Y(x, y)$ is the analytic cocycle satisfying the equation defined by M in the basis \mathbf{e} . Then the analytic cocycle associated to the equation defined by M in the basis $H(x)\mathbf{e}$ is given by*

$$H(x)Y(x, y)H(y)^{-1} \quad (3.10)$$

Proof. One verifies easily that $H(x)Y(x, y)H(y)^{-1}$ is a K -rational analytic cocycle, and that it is solution of the new equation in the new base $H(x)\mathbf{e}$. \square

COROLLARY 3.10. *The radius $\text{Rad}(Y(x, y), |\cdot|_*)$ is invariant under $\mathcal{H}_K(X)$ -base changes.*

Proof. The radius $\text{Rad}(Y(x, y_0))$ of $Y(x, y_0)$, as function of x , is not invariant under base changes by matrices $H(x) \in GL_n(\mathcal{H}_K(X))$, as one can see from equation (3.10). Indeed the matrix $H(x)$ may have a small radius with respect to that of $Y(x, y_0)$. By equation (2.11) we notice that $H(x)$ converges at least on the disk $D^-(y_0, \rho_{y_0, X})$, hence $\text{Rad}(Y(x, y), y_0) := \min(\text{Rad}(Y(x, y_0)), \rho_{y_0, X})$ is invariant under $\mathcal{H}_K(X)$ -base changes. \square

LEMMA 3.11 (Tensor product). *Consider two K -rational analytic cocycles $Y_1(x, y) = (y_{i,j}^{(1)})_{i,j=1,\dots,n_1}$, and $Y_2(x, y) = (y_{l,k}^{(2)})_{l,k=1,\dots,n_2}$ on X . Define $y_{(i,l),(j,k)}^{(3)} := y_{i,j}^{(1)} \cdot y_{l,k}^{(2)}$. Then*

$$Y_3(x, y) := (y_{(i,l),(j,k)}^{(3)})_{(i,l),(j,k) \in [1,n_1] \times [1,n_2]} \quad (3.11)$$

is a K -rational analytic cocycle. We call $Y_3(x, y)$ the tensor product of $Y_1(x, y), Y_2(x, y)$.

Proof. If \mathcal{U}_i is the uniform neighborhood of the diagonal over which Y_i converges, then clearly $\mathcal{U}_3 := \mathcal{U}_1 \cap \mathcal{U}_2$ is uniform, moreover $Y_3(x, x) = \text{Id}$, and $Y_3(x, y)$ converges on \mathcal{U}_3 . It is enough to show that $Y_3(x, y)Y_3(y, z) = Y_3(x, z)$ for all $(x, y)(y, z)(x, z) \in \mathcal{U}_3$. We know that $Y_1(x, y), Y_2(x, y)$ are Taylor solutions of two differential equations. For $s = 1, 2$ let M_s be the differential modules of dimension n_s defined by the equations $d/dx(Y_s(x, y)) = G_s(x)Y_s(x, y)$. If $e_1^{(s)}, \dots, e_{n_s}^{(s)} \in M_s$ is the basis of M_s for which $Y_s(x, y)$ is the solution, then the columns of $Y_s(x, y)$ defines a basis of solutions with values in $\mathcal{A}_K(y, R)$: $\alpha_1^{(s)}, \dots, \alpha_{n_s}^{(s)} : M_s \rightarrow \mathcal{A}_K(y, R)$, with $y_{i,j}^{(s)} = \alpha_j^{(s)}(e_i)$. We interpret $Y_3(x, y)$ as the Taylor solution of the tensor product $M_1 \otimes M_2$. Hence this will prove the required cocycle property of $Y_3(x, y)$, thanks to the fact that $Y_3(x, y)$ is a solution of a differential equation. Indeed $\alpha_{j,k}^{(3)} := \alpha_j^{(1)} \otimes \alpha_k^{(2)}$ is a basis of the K -vector space of solutions of $M_3 := M_1 \otimes M_2$. Choosing the basis $e_{i,l}^{(3)} := e_i^{(1)} \otimes e_l^{(2)}$ one has $\alpha_{j,k}^{(3)}(e_{i,l}^{(3)}) = y_{i,j}^{(1)} y_{l,k}^{(2)} = y_{(i,l),(j,k)}^{(3)}$. By the uniqueness of the Taylor solution $Y_3(x, y)$ is the Taylor solution of $M_1 \otimes M_3$ in the basis $\{e_{i,l}^{(3)}\}_{(i,l) \in [1,n_1] \times [1,n_2]}$. \square

3.4 Elementary stratifications

Let X be an affinoid, and let $R > 0$. An *(elementary) R -stratification on X* (M, χ_M) over X is a finite free $\mathcal{H}_K(X)$ -module M together with an isomorphism

$$\chi_M : p_1^* M|_{\mathcal{T}(X,R)} \xrightarrow{\sim} p_2^* M|_{\mathcal{T}(X,R)}, \quad (3.12)$$

where $p_i : X \times X \rightarrow X$ denotes the projection on the i -th factor. We assume moreover that:

- i) If $p_{i,j} : X \times X \times X \rightarrow X \times X$ denotes the projection on the (i,j) -factor, then

$$p_{1,2}^*(\chi_M) \circ p_{2,3}^*(\chi_M) = p_{1,3}^*(\chi_M) \quad (3.13)$$

on $p_{1,2}^{-1}(\mathcal{T}(X,R)) \cap p_{2,3}^{-1}(\mathcal{T}(X,R)) \cap p_{1,3}^{-1}(\mathcal{T}(X,R))$.

- ii) If $\Delta : X \rightarrow X \times X$ denotes the diagonal immersion, then $\Delta^*(\chi_M) = \text{Id}_M$ on M (here we identify canonically M with $\Delta^* p_i^* M$).

If a basis of M is chosen and if $Y(x,y) \in GL_{n \times n}(\mathcal{H}_K(\mathcal{T}(X,R)))$ denotes the matrix of χ_M in this basis then $Y(x,y)$ is an analytic cocycle.

A morphism $(M, \chi_M) \rightarrow (N, \chi_N)$ of regular stratifications is a $\mathcal{H}_K(X)$ -linear map $\alpha : M \rightarrow N$ satisfying

$$(p_2^* \alpha)|_{\mathcal{T}(X,R)} \circ \chi_M = \chi_N \circ (p_1^* \alpha)|_{\mathcal{T}(X,R)}. \quad (3.14)$$

We denote by $\text{Hom}^X(M, N)$ the K -vector space of morphisms. We call $\text{Strat}(\mathcal{H}_K(X))^{[R]}$ the category of (elementary) R -stratifications.

3.4.1 If $d - \text{Mod}(\mathcal{H}_K(X))^{[R]}$ denotes the fully faithful sub-category of $d - \text{Mod}(\mathcal{H}_K(X))$ whose objects have a Taylor solution converging on $\mathcal{T}(X,R)$, then we have a functor

$$S : d - \text{Mod}(\mathcal{H}_K(X))^{[R]} \xrightarrow{\sim} \text{Strat}(\mathcal{H}_K(X))^{[R]} \quad (3.15)$$

defined as follows: $S(M, \nabla^M)$ is the R -stratification (M, χ_M) defined by the Cauchy solution $Y(x,y)$ of M . For all morphisms $\alpha : M \rightarrow N$ one has $S(\alpha) := \alpha$. It follows essentially from section 3.3 that this is a well defined functor. In terms of matrices the well definition follows from the fact that $H(z)$ is the matrix of a morphisms $\alpha : (M, \nabla^M) \rightarrow (N, \nabla^N)$ if and only if $H(z)Y_N(z,w) = Y_M(z,w)H(w)$ for all $(z,w) \in \mathcal{T}(X,R)$, where $Y_M(z,w)$ and $Y_N(z,w)$ are the analytic cocycles defined by M and N in the fixed bases. This proves that α verifies (3.14).

Conversely we have a functor in the other direction

$$D : \text{Strat}(\mathcal{H}_K(X))^{[R]} \xrightarrow{\sim} d - \text{Mod}(\mathcal{H}_K(X))^{[R]} \quad (3.16)$$

defined as follows: $D(M, \chi_M)$ is the unique differential module defined by proposition 3.5, and D is the identity on the morphisms: $D(\alpha) = \alpha$. The following theorem follows essentially from section 3.3

THEOREM 3.12. *The functor $S : d - \text{Mod}(\mathcal{H}_K(X))^{[R]} \xrightarrow{\sim} \text{Strat}(\mathcal{H}_K(X))^{[R]}$ is an equivalence of categories, with quasi inverse D . If $\text{Strat}(\mathcal{H}_K(X))$ denotes the union $\bigcup_{R>0} \text{Strat}(\mathcal{H}_K(X))^{[R]}$, then the functor S (resp. D) commutes with the inclusions of categories and defines an equivalence $S : d - \text{Mod}(\mathcal{H}_K(X)) \xrightarrow{\sim} \text{Strat}(\mathcal{H}_K(X))$, with quasi inverse D . \square*

4. Generalized Σ -difference equations

4.1 Generalized Σ -algebras and generalized Σ -modules

4.1.1 *Generalized Σ -algebras.* A *generalized σ -algebra* is a ring B together with a collection of pairs of ring morphisms

$$\{ B \xrightarrow[\iota_\sigma]{\sigma} B_\sigma \}_{\sigma \in \Sigma}, \quad (4.1)$$

where $\{B_\sigma\}_{\sigma \in \Sigma}$ is a family of rings.

4.1.2 Σ -constants. We denote by B^Σ the subring of B of elements $b \in B$ verifying $\sigma(b) = i_\sigma(b)$, for all $\sigma \in \Sigma$. We call its elements Σ -constants of B or equivalently Σ -invariant elements.

4.1.3 Generalized Σ -modules. A *generalized Σ -module* is a finite free B -module M together with the data of a collection of isomorphisms

$$\{ \sigma^M : \sigma^* M \xrightarrow{\sim} i_\sigma^* M \}_{\sigma \in \Sigma}, \quad (4.2)$$

where $\sigma^* M$ and $i_\sigma^* M$ denote the scalar extension of M to B_σ via σ and i_σ respectively. A morphism between two generalized Σ -modules is a B -linear map $\alpha \in \text{Hom}_B(M, N)$ satisfying $i_\sigma^*(\alpha) \circ \sigma^M = \sigma^N \circ \sigma^*(\alpha)$, for all $\sigma \in \Sigma$, as expressed by the following diagram:

$$\begin{array}{ccc} \sigma^* M & \xrightarrow{\sigma^M} & i_\sigma^* M \\ \sigma^*(\alpha) \downarrow & \circlearrowleft & \downarrow i_\sigma^*(\alpha) \\ \sigma^* N & \xrightarrow{\sigma^N} & i_\sigma^* N \end{array} \quad . \quad (4.3)$$

We denote by $\text{Hom}_B^\Sigma(M, N)$ the set of morphisms, it is canonically a B^Σ -module. The category of generalized Σ -modules will be denoted by $\Sigma\text{-Mod}(B)$. This is a tensor category. The unit object is $\mathbb{I}_B := (B, \sigma^B)$, with $\sigma^B = \text{Id}_{B_\sigma}$, where we identify $B \otimes_{B, \sigma} B_\sigma$ and $B \otimes_{B, i_\sigma} B_\sigma$ with B_σ . The ring of endomorphisms of the unit object is then canonically identified with B^Σ . The internal \otimes is given by $(M \otimes_B N, \sigma^{M \otimes N})$, with $\sigma^{M \otimes N}$ defined by

$$\sigma^{M \otimes N} = \sigma^M \otimes \sigma^N, \quad (4.4)$$

where we identify $(M \otimes_B N) \otimes_{B, *} B_\sigma$ with $(M \otimes_{B, *} B_\sigma) \otimes_{B_\sigma} (N \otimes_{B, *} B_\sigma)$, where $*$ denotes σ or i_σ .

REMARK 4.1. If Σ is reduced to a single element σ , and if $B = B_\sigma$, and $i_\sigma = \text{Id}$, then we obtain the classical definition of σ -modules.

4.1.4 Matrix of σ^M . If $\mathbf{e} := \{e_1, \dots, e_n\} \subset M$ is a basis of M , the operator σ^M is given by its values on $\mathbf{e} \otimes 1$. If $\sigma^M(e_i \otimes 1) = \sum_{j=1}^n a_{\sigma, i, j}(e_j \otimes 1)$ we call $A_\sigma := (a_{\sigma, i, j})_{i, j=1, \dots, n} \in GL_n(B_\sigma)$ the matrix of σ^M in the basis \mathbf{e} .

4.2 Solutions (formal definition)

By analogy with the differential setting, we shall define now the notion of solution of a generalized Σ -module.

4.2.1 Extension of generalized Σ -algebras. Let Σ' be a family of morphisms, and let $\{B' \xrightarrow[i_{\sigma'}]{\sigma'} B'_{\sigma'}\}_{\sigma' \in \Sigma'}$ be a generalized Σ' -algebra. Assume given a surjective map $\mathcal{P} : \Sigma' \rightarrow \Sigma$. Assume moreover that B' is a B -algebra with structural morphism $j : B \rightarrow B'$, and assume given, for all $\sigma' \in \Sigma'$, a ring morphism $\Delta_{\sigma'} : B_{\mathcal{P}(\sigma')} \rightarrow B_{\sigma'}$ making $B_{\sigma'}$ a $B_{\mathcal{P}(\sigma')}$ -algebra. We say that the data of

$$(B' \xrightarrow[i_{\sigma'}]{\sigma'} B'_{\sigma'}, j, \Delta_{\sigma'})_{\sigma' \in \Sigma'} \quad (4.5)$$

is an extension of generalized Σ -algebras if the following diagrams commute for all $\sigma' \in \Sigma'$:

$$\begin{array}{ccc} B' & \xrightarrow{\sigma'} & B'_{\sigma'} \\ j \uparrow & \circlearrowleft & \uparrow \Delta_{\sigma'} \\ B & \xrightarrow{\sigma} & B_{\sigma} \end{array} \quad ; \quad \begin{array}{ccc} B' & \xrightarrow{i_{\sigma'}} & B'_{\sigma'} \\ j \uparrow & \circlearrowleft & \uparrow \Delta_{\sigma'} \\ B & \xrightarrow{i_{\sigma}} & B_{\sigma} \end{array} , \quad (4.6)$$

where $\sigma = \mathcal{P}(\sigma')$.

4.2.2 Pull back. With the above notations let $M \in \Sigma - \text{Mod}(B)$. The B' -module $j^*M := M \otimes_{B,j} B'$ is a Σ' -module in a canonical way by setting $(\sigma')^{j^*M} := j_{\sigma'}^*(\sigma^M) = \sigma^M \otimes_{B_{\sigma}, \Delta_{\sigma'}} \text{Id}_{B'_{\sigma'}}$, with $\sigma = \mathcal{P}(\sigma')$, where $(\sigma')^* j^*M$ (resp. $i_{\sigma'}^* j^*M$) is canonically identified with $\Delta_{\sigma'}^* \sigma^*M$ (resp. $\Delta_{\sigma'}^* i_{\sigma}^*M$). If $A_{\sigma} = (a_{\sigma,i,j})_{i,j} \in GL_n(B_{\sigma})$ is the matrix of σ^M in a fixed basis $\mathbf{e} \subset M$ (cf. Section 4.1.4), then the matrix of $(\sigma')^{j^*M}$ in the basis $\mathbf{e} \otimes 1$ is given by

$$A_{\sigma'} := \Delta_{\sigma'}(A_{\sigma}) = (\Delta_{\sigma'}(a_{\sigma,i,j}))_{i,j} \in GL_n(B'_{\sigma'}). \quad (4.7)$$

If $\alpha : M \rightarrow N$ is a morphism in $\Sigma - \text{Mod}(B)$, then $j^*\alpha : j^*M \rightarrow j^*N$ commutes with Σ' and lies hence in $\Sigma' - \text{Mod}(B')$.

The definition of the scalar extension functor depends highly on the chosen family $\Delta := \{\Delta_{\sigma'}\}_{\sigma' \in \Sigma'}$, so we denote it by

$$j_{\Delta}^* : \Sigma - \text{Mod}(B) \longrightarrow \Sigma' - \text{Mod}(B'). \quad (4.8)$$

4.2.3 Solutions. Let $M \in \Sigma - \text{Mod}(B)$. A *solution of M with values in B'* is a B -linear map $\alpha : M \rightarrow B'$ satisfying $(\sigma')^* \alpha \otimes 1 = i_{\sigma'}^*(\alpha \otimes 1) \circ \Delta_{\sigma'}^* \sigma^M$, for all $\sigma' \in \Sigma'$, as expressed by the following commutative diagram:

$$\begin{array}{ccccccc} \Delta_{\sigma'}^* \sigma^*M & \xrightarrow{\sim} & (\sigma')^* j^*M & \xrightarrow{(\sigma')^* j^* \alpha} & (\sigma')^*(B') & \xrightarrow{\sim} & B'_{\sigma'} \\ \sigma^{j^*M} := \Delta_{\sigma'}^* \sigma^M \downarrow & & & \circlearrowleft & & & \parallel \\ \Delta_{\sigma'}^* i_{\sigma}^*M & \xrightarrow{\sim} & i_{\sigma'}^* j^*M & \xrightarrow{i_{\sigma'}^* j^* \alpha} & i_{\sigma'}^*(B') & \xrightarrow{\sim} & B'_{\sigma'} \end{array} \quad (4.9)$$

where $\sigma = \mathcal{P}(\sigma')$. If \mathbb{I}_B denotes the unit object, then the set of solutions is canonically identified with $\text{Hom}_{B'}^{\Sigma'}(j_{\Delta}^*M, \mathbb{I}_{B'})$. The solutions of M with values in B' are hence naturally a $(B')^{\Sigma'}$ -module.

4.2.4 Matrix notation. Fix a $\sigma' \in \Sigma'$, and let $\sigma = \mathcal{P}(\sigma')$. If A_{σ} denotes the matrix of σ^M in the basis $\mathbf{e} = \{e_1, \dots, e_n\} \subset M$, then the vector $\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in (B')^n$, defined by $y_i := \alpha(e_i)$, verifies $\begin{pmatrix} \sigma'(y_1) \\ \vdots \\ \sigma'(y_n) \end{pmatrix} = A_{\sigma'} \cdot \begin{pmatrix} i_{\sigma'}(y_1) \\ \vdots \\ i_{\sigma'}(y_n) \end{pmatrix}$, where $A_{\sigma'} = \Delta_{\sigma'}(A_{\sigma}) \in GL_n(B_{\sigma'})$. By abuse of notation, we will say that M is defined in the basis \mathbf{e} by the family of Σ -difference equations

$$\sigma(Y) = A_{\sigma} \cdot i_{\sigma}(Y), \quad \sigma \in \Sigma. \quad (4.10)$$

4.3 Stratifications as generalized Σ -difference equations

We introduce the following algebra

$$\mathcal{A}_K(\Delta^{\dagger}) := \cup_{0 < R \leq r_X} \mathcal{A}_K(\mathcal{T}(X, R)). \quad (4.11)$$

4.3.1 The category $\text{Strat}(\mathcal{H}_K(X))$ can be understood as a category of generalized Σ -difference equations, with $B := \mathcal{H}_K(X)$, $B_\sigma := \mathcal{A}_K(\Delta^\dagger)$, $\Sigma_{\text{Strat}} := \{\sigma := p_1 : \mathcal{H}_K(X) \rightarrow \mathcal{A}_K(\Delta^\dagger)\}$ is reduced to the single element induced by the first projection $p_1 : X \times X \rightarrow X$, and $i_\sigma = p_2$ is induced by the second projection.

4.3.2 We consider now on $B' := \mathcal{A}_K(\Delta^\dagger)$ the Σ_{Strat} -algebra structure given by $\Sigma'_{\text{Strat}} = \Sigma_{\text{Strat}}$, $\mathcal{P} := \text{Id}_{\Sigma_{\text{Strat}}}$, $B'_\sigma := \mathcal{A}_K(\Delta^\dagger)$, $\sigma = \text{Id}_{\mathcal{A}_K(\Delta^\dagger)}$, and $i_\sigma := \text{Tw} : \mathcal{A}_K(\Delta^\dagger) \xrightarrow{\sim} \mathcal{A}_K(\Delta^\dagger)$, where Tw is the twist map $f(x, y) \mapsto f(y, x)$.

Then $(\mathcal{A}_K(\Delta^\dagger), \sigma := \text{Id}, i_\sigma := \text{Tw})$ is canonically a generalized Σ_{Strat} -algebra over $(\mathcal{H}_K(X), \Sigma_{\text{Strat}})$ via the morphisms $j = p_1 : \mathcal{H}_K(X) \rightarrow \mathcal{A}_K(\Delta^\dagger)$ and $\Delta_\sigma := \text{Id}$:

$$\begin{array}{ccc} \mathcal{A}_K(\Delta^\dagger) & \xrightarrow{\text{Id}} & \mathcal{A}_K(\Delta^\dagger) \\ p_1 \uparrow & \odot & \uparrow \text{Id} \\ \mathcal{H}_K(X) & \xrightarrow{p_1} & \mathcal{A}_K(\Delta^\dagger) \end{array} \quad ; \quad \begin{array}{ccc} \mathcal{A}_K(\Delta^\dagger) & \xrightarrow{\text{Tw}} & \mathcal{A}_K(\Delta^\dagger) \\ p_1 \uparrow & \odot & \uparrow \text{Id} \\ \mathcal{H}_K(X) & \xrightarrow{p_2} & \mathcal{A}_K(\Delta^\dagger) \end{array} . \quad (4.12)$$

REMARK 4.2. We could consider another generalized Σ_{Strat} -algebra structure on $\mathcal{A}_K(\Delta^\dagger)$ given by $\sigma := \text{Tw}$, $i_\sigma := \text{Id}$, $j := p_2$, $\Delta_\sigma := \text{Id}_{\mathcal{A}_K(\Delta^\dagger)}$. One sees easily that $(\mathcal{A}_K(\Delta^\dagger), \sigma := \text{Id}, i_\sigma := \text{Tw})$ and $(\mathcal{A}_K(\Delta^\dagger), \sigma := \text{Tw}, i_\sigma := \text{Id})$ are canonically isomorphic as $(\mathcal{H}_K, \Sigma_{\text{Strat}})$ -algebras.

4.3.3 *Analytic cocycles as solutions of generalized Σ_{Strat} -equations.* It is clear that every elementary stratification (M, χ_M) is trivialized by the generalized Σ_{Strat} -algebra $(\mathcal{A}_K(\Delta^\dagger), \sigma = \text{Id}, i_\sigma = \text{Tw})$. Namely if (M, χ_M) is an elementary stratification, and if $Y(x, y) \in GL_n(\mathcal{A}_K(\mathcal{T}(X, R)))$ is its co-cycle in a basis. Then $Y(x, y)$ is a complete basis of solutions of M with values in $B' := \mathcal{A}_K(\mathcal{T}(X, R))$.

5. Deformation functor

In this section we will define the Σ -deformation functor for a class of automorphisms of X called *infinitesimal*.

5.1 Infinitesimal automorphisms of $\mathcal{H}_K(X)$

Let X be an affinoid and let $\sigma : X \xrightarrow{\sim} X$ be an automorphism. We denote again by σ the induced continuous automorphism of $\mathcal{H}_K(X)$, one has $\sigma(f(T)) := f(\sigma(T))$. For a complete field Ω/K , and for a point $y \in X(\Omega)$, if there exists $R \leq \rho_{y,X}$ such that, for all $x \in D^-(y, \rho_{y,X})$ one has

$$|\sigma(x) - x|_\Omega < R \leq \rho_{x,X} (= \rho_{y,X}), \quad (5.1)$$

then the action of σ extends uniquely to a continuous endomorphism of $\mathcal{A}_\Omega(y, R)$. We are hence interested to find solutions of equations of the form (4.10) with values in $\mathcal{A}_\Omega(y, R)$. We will call these solution *Taylor solutions (at y)* of the equation (4.10). We set

$$\delta_\sigma(T) := \sigma(T) - T. \quad (5.2)$$

DEFINITION 5.1. We call *infinitesimal automorphism of $\mathcal{H}_K(X)$* a ring automorphism $\sigma : \mathcal{H}_K(X) \xrightarrow{\sim} \mathcal{H}_K(X)$ satisfying:

- i) σ is a K -linear *continuous* automorphism of $\mathcal{H}_K(X)$,
- ii) For all complete valued field Ω , and all $y \in X(\Omega)$, σ satisfies

$$|\delta_\sigma(y)|_\Omega < \rho_{y,X}. \quad (5.3)$$

We denote by \mathfrak{S}_X the family of infinitesimal automorphisms of X .

- REMARK 5.2. i) The family \mathfrak{S}_X is stable by composition.
 ii) Since $H_K^{\text{rat}}(X)$ is dense in $\mathcal{H}_K(X)$, we notice that condition i) of the above Definition implies $\sigma(f(T)) = f(\sigma(T))$ for all $f \in \mathcal{H}_K(X)$.
 iii) By Sections 2.1 and 2.3, one sees that condition ii) is equivalent to ask that

$$|\delta_\sigma(T)|_* < \rho_{|\cdot|_*, X}, \quad \text{for all } |\cdot|_* \in \mathcal{M}(X). \quad (5.4)$$

LEMMA 5.3 (Infinitesimality condition). *Let $X := D^+(c_0, R_0) - \cup_{i=1}^n D^-(c_i, R_i)$. Assume that σ satisfies condition i) of Definition 5.1. The following conditions are equivalent:*

- i) The condition (5.4) is verified for all element of $\mathcal{M}(X)$
 ii) The condition (5.4) is verified on the elements of its maximal Skeleton \mathcal{S}_X . More explicitly σ is infinitesimal if and only if for all $i = 1, \dots, n$ one has (cf. section 2.5, and Prop. 2.12)

$$\delta_\sigma(|\cdot|_{c_i, \rho}) < \rho, \quad \text{for all } R_i \leq \rho \leq R_0. \quad (5.5)$$

- iii) The condition (5.4) is verified for the critical points \mathcal{P}_X of the maximal Skeleton \mathcal{S}_X of $\mathcal{M}(X)$. More precisely σ is infinitesimal if and only if

$$\begin{cases} |\delta_\sigma(T)|_{c_i, R_i} < R_i, & \text{for all } i = 0, 1, \dots, n, \text{ (Shilov Boundary)} \\ |\delta_\sigma(T)|_{c_i, |c_i - c_j|} < |c_i - c_j|, & \text{for all } i, j = 1, \dots, n, i \neq j \text{ (Critical points)}. \end{cases} \quad (5.6)$$

Proof. Clearly i) \implies ii) \implies iii). Now ii) implies i) because for all $|\cdot|_* \in \mathcal{M}(X)$, there exists $|\cdot|_{**} \in \mathcal{S}_X$ such that $|\cdot|_* \leq |\cdot|_{**}$. Hence one has $\delta_\sigma(|\cdot|_*) \leq \delta_\sigma(|\cdot|_{**})$, and by Lemma 2.6 one has also $\rho_{|\cdot|_*, X} = \rho_{|\cdot|_{**}, X}$. Now iii) implies ii) because by Proposition 2.12, the value of $\rho_{|\cdot|_*, X}$ have been explicitated for all $|\cdot|_*$ in the maximal Skeleton \mathcal{S}_X , it is hence sufficient to apply Proposition 2.13, by the logarithmic properties of the function $\rho \mapsto |\delta_\sigma(T)|_{c_i, \rho}$ for $\rho \in [R_i, R_0]$. \square

5.2 Σ -compatibility

In this section we introduce the algebra of Σ -compatible functions which will be the key notion for the definition of the deformation functor.

5.2.1 \mathfrak{S} -compatible functions. Let as usual $X = D^+(c_0, R_0) - \cup_{i=1}^n D^-(c_i, R_i)$. Let $f(x, y)$ be a convergent function on $\mathcal{T}(X, R)$, for some $R > 0$.

DEFINITION 5.4. Let $\mathfrak{S} \subseteq \mathcal{H}_K(X)$ be a family of functions on X . We say that $f(x, y)$ is \mathfrak{S} -compatible if for all complete valued field extension Ω/K , for all $x \in X(\Omega)$, and for all $g(x) \in \mathfrak{S}$, one has

$$|g(x)|_\Omega < \text{Rad}(f(x, y), |\cdot|_x). \quad (5.7)$$

We denote by $\mathcal{A}_K(\mathfrak{S})$ the sub-algebra of $\bigcup_{R>0} \mathcal{A}_K(\mathcal{T}(X, R))$ of \mathfrak{S} -compatible functions.

PROPOSITION 5.5 (Compatibility condition). *The function $f(x, y)$ is \mathfrak{S} -compatible if and only if one of the following equivalent conditions is fulfilled:*

- i) For all $g \in \mathfrak{S}$ one has

$$|g(x)|_* < \text{Rad}(f(x, y), |\cdot|_*) , \quad \text{for all } |\cdot|_* \in \mathcal{M}(X). \quad (5.8)$$

- ii) If $\mathcal{S}_X = \cup_{i=1}^n \{ |\cdot|_{c_i, \rho} \}_{\rho \in [R_i, R_0]}$ is the maximal Skeleton of $\mathcal{M}(X)$, then for all $g \in \mathfrak{S}$ one has

$$|g(T)|_{c_i, \rho} < \text{Rad}(f(x, y), |\cdot|_{c_i, \rho}) , \quad \text{for all } |\cdot|_{c_i, \rho} \in \mathcal{S}_X. \quad (5.9)$$

- iii) If $\mathcal{P}_X = \{ |\cdot|_{c_i, R_i} \}_{i=1, \dots, n} \cup \{ |\cdot|_{c_i, |c_i - c_j|} \}_{i, j=1, \dots, n}$ is the set of critical points, then for all $g \in \mathfrak{S}$

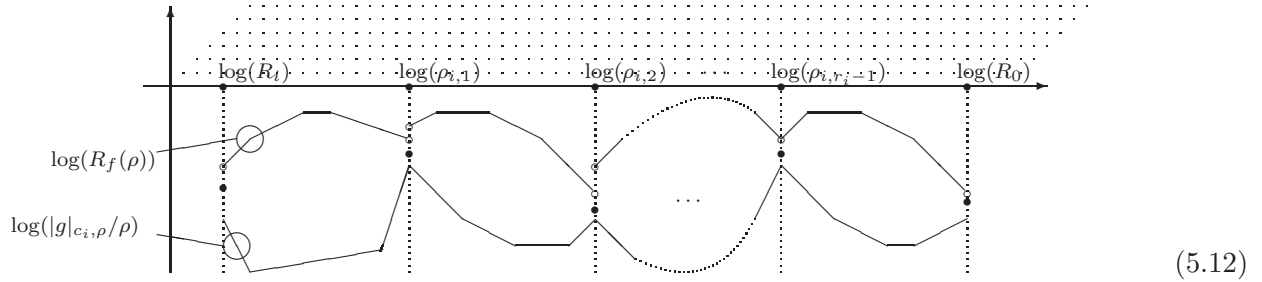
one has

$$\begin{cases} |g(T)|_{c_i, R_i} < \text{Rad}(f(x, y), |\cdot|_{c_i, R_i}), & \text{for all } i = 0, \dots, n, \\ |g(T)|_{c_i, |c_i - c_j|} < \text{Rad}(f(x, y), |\cdot|_{c_i, |c_i - c_j|}), & \text{for all } i, j = 1, \dots, n, i \neq j. \end{cases} \quad (5.10)$$

Proof. The first assertion is clearly equivalent to the compatibility condition (cf. Section 2.1), and the implications i) \implies ii) \implies iii) are clear. Assume then that condition (5.8) holds for every point of \mathcal{S}_X . If $|\cdot|_* \in \mathcal{M}(X)$ there exists $|\cdot|_{**} \in \mathcal{S}_X$ satisfying $|\cdot|_* \leq |\cdot|_{**}$, then

$$|g|_* \leq |g|_{**}, \quad \text{and} \quad \text{Ray}(f(x, y), |\cdot|_*) \geq \text{Ray}(f(x, y), |\cdot|_{**}), \quad (5.11)$$

by Transfer principle (cf. Lemma 2.17). This proves the equivalence between i) and ii). The implication iii) \implies ii) follows from Proposition 2.13 applied to the function g , and Proposition 2.18. The situation is expressed by the following log-graphic expressing (with the notations of Propositions 2.13 and 2.18) the functions $\log(\rho) \mapsto R_f(\rho) := \log(\text{Rad}(f(x, y), |\cdot|_{c_i, \rho})/\rho)$ and $\log(\rho) \mapsto \log(|g(T)|_{c_i, \rho}/\rho)$:



This proves the Proposition. \square

5.2.2 The algebra $\mathcal{A}_K(\Sigma)$. Let $\Sigma \subseteq \mathfrak{S}_X$ be a family of infinitesimal operators on X . Let as usual $X = D^+(c_0, R_0) - \cup_{i=1}^n D^-(c_i, R_i)$. We denote by $\mathcal{A}_K(\Sigma)$ the sub-algebra of $\mathcal{A}_K(\Delta^\dagger)$ (cf. (4.11)) whose elements are \mathfrak{S} -compatible functions, where

$$\mathfrak{S} := \{ \delta_\sigma(T) \mid \sigma \in \Sigma \}. \quad (5.13)$$

The algebra $\mathcal{A}_K(\Sigma)$ is canonically endowed with the morphisms $p_1, p_2 : \mathcal{H}_K(X) \rightarrow \mathcal{A}_K(\Sigma)$ induced by the two projections $p_i : X \times X \rightarrow X$. The main proposition of this section is the following.

PROPOSITION 5.6. *For all $\sigma \in \Sigma$, the morphism $\Delta_\sigma : \mathcal{A}_K(\Sigma) \rightarrow \mathcal{H}_K(X)$ sending a function $f(x, y) \mapsto f(\sigma(x), x)$ is well defined.*

Proof. We have to show that if $f(x, y) = \sum_{n \geq 0} a_n(y)(x - y)^n \in \mathcal{A}_K(\Sigma)$, then the series of functions defining $f(\sigma(x), x) = \sum_n a_n(x) \delta_\sigma(x)^n$ converges and belongs to $\mathcal{H}_K(X)$. We need the following

LEMMA 5.7 (Reduction to tubular σ -admissibility). *Let X be an affinoid, and let σ be an infinitesimal automorphism of X . Let $f(x, y) \in \mathcal{A}_K(\sigma)$. Then there exists an admissible covering $X = \cup_{r=0, \dots, m} X_r$, such that, for all $r = 0, \dots, m$, one has*

$$\|\delta_\sigma\|_{X_r} < \text{Rad}(f(x, y), X_r), \quad (5.14)$$

where $\text{Rad}(f(x, y), X_r)$ denotes the global radius of (the restriction of) $f(x, y)$ on X_r (cf. Section 2.7.1). In particular $f(x, y)$ converges on a tubular neighborhood of $\mathcal{T}(X_r, R)$ with $\|\delta_\sigma\|_{X_r} < R$.

Proof. We preserve the notations of Proposition 5.5. We define firstly a covering of the maximal Skeleton \mathcal{S}_X . For all fixed index $i \in \{0, \dots, n\}$, there exists a finite number of open intervals $I_{i,1}, \dots, I_{i,s}$ such that $[R_i, R_0] = \cup_{h=1}^{k_i} I_{i,h}$ (we are assuming that $I_{i,1} = [R_i, \varepsilon_i[$, and $I_{i,k_i} =]\varepsilon'_i, R_0]$), and having the following properties:

- i) For all $h = 1, \dots, k_i$, there exists $R_{i,h} > 0$ such that $|\delta_\sigma|_{c_i,\rho} < R_{i,h} < \text{Rad}(f(x,y), |\cdot|_{c_i,\rho})$, for all $\rho \in I_{i,h}$;
- ii) None of the borders of $I_{i,h}$ is a critical point, and every $I_{i,h}$ contains at most a single critical point;
- iii) Let $|\cdot|_*$ be a critical point, let $I_{i_1,s_1}, \dots, I_{i_q,s_q}$ be the intervals passing through $|\cdot|_*$, and let $\mathcal{S}(|\cdot|_*) \subset \mathcal{S}_X$ be the (non disjoint) union of $I_{i_1,s_1}, I_{i_2,s_2}, \dots, I_{i_q,s_q}$ in \mathcal{S}_X (cf. picture after Def. 2.8). Then we assume that $I_{i_1,s_1}, \dots, I_{i_q,s_q}$ are sufficiently small to guarantee the existence of a number $R > 0$ such that $|\delta_\sigma|_{**} < R < \text{Rad}(f(x,y), |\cdot|_{**})$, for all $|\cdot|_{**}$ belonging to $\mathcal{S}(|\cdot|_*)$.

The existence of such a covering of \mathcal{S}_X follows immediately from Proposition 5.5.

Now define $Z_{i,h} := \mathcal{C}(c_i, I_{i,h}^+) \cap X$, where $I_{i,h}^+$ denotes the closure of $I_{i,h}$. The maximal Skeleton of the affinoid $Z_{i,h}$ is a piece of the maximal Skeleton of X because we have not added any new holes, we have simply enlarged the holes of X (cf. section 2.5). We define then the family $\{X_r\}_r$ as follows.

If $I_{i,h}$ does not contain any critical point, then we include $Z_{i,h}$ into the family $\{X_r\}_r$. Notice that in this case $Z_{i,h}$ coincides with annulus $\mathcal{C}(c_i, I_{i,h}^+)$ which is contained in X . Moreover its maximal Skeleton is reduced to the interval $I_{i,h}^+$. In particular, by the property i), $Z_{i,h}$ satisfies the property (5.14), because of Proposition 5.5. It remains to “cover” the critical points. With the above notations, for all critical point $|\cdot|_*$ we include in the family $\{X_r\}_r$ the sub-affinoid $X(|\cdot|_*) := Z_{i_1,h_1} \cap \dots \cap Z_{i_q,h_q}$. It is clear that now the family $\{X_r\}_r$ is an admissible covering of X , and that the maximal Skeleton of $X(|\cdot|_*)$ is given by $\mathcal{S}(|\cdot|_*)$. By Proposition 5.5, the sub-affinoid $X(|\cdot|_*)$ so defined verifies (5.14) because of the assumption iii). \square

Continuation of the proof of Proposition 5.6 : We claim that, if $f(x,y) = \sum_{n \geq 0} a_n(y)(x - y)^n$, then the series of functions $\Delta_\sigma f(x) := \sum_{n \geq 0} a_n(x) \delta_\sigma(x)^n$ converges on X because of the σ -compatibility property of $f(x,y)$. Indeed, by Lemma 5.7, we are reduced to check the convergence on every single affinoid X_r of the covering. But X_r has the property that $\|\delta_\sigma\|_{X_r} < \text{Rad}(f(x,y), X_r)$, so $\lim_n \|a_n \delta_\sigma\|_{X_r} \leq \lim_n \|a_n\|_{X_r} \|\delta_\sigma\|_{X_r} = 0$ since $f(x,y)$ converges on $\mathcal{T}(X_r, R)$, with $R = \text{Rad}(f(x,y), X_r)$ (cf. (2.40)). Hence the matrix $\Delta_\sigma f(x)$ has coefficients in $\mathcal{H}_K(X)$. \square

5.3 The Deformation functor

The aim of this section is to define the Σ -deformation of a differential module as the pull-back (in the sense of section 4.2.2) of its elementary stratification via the morphism $\Delta_\sigma : \mathcal{A}_K(\Sigma) \rightarrow \mathcal{H}_K(X)$.

5.3.1 Analytic Cocycles as solutions of σ -difference equations By section 3.1 the Taylor solution of a differential equation is an analytic cocycle, and reciprocally (cf. Proposition 3.5). The aim of this sub-section is to prove that every Σ -compatible analytic cocycle is solution of an unique σ -difference equation, for all $\sigma \in \Sigma$.

COROLLARY 5.8. *Let $\Sigma \subseteq \mathfrak{S}_X$ be a family of infinitesimal automorphisms of $\mathcal{H}_K(X)$. Let $Y(x,y)$ be a K -rational analytic cocycle. If $Y(x,y)$ is Σ -compatible, then, for all $\sigma \in \Sigma$, $Y(x,y)$ is solution of a unique σ -difference equation with coefficients in $\mathcal{H}_K(X)$ of the type*

$$Y(\sigma(x), y) = A_\sigma(x) \cdot Y(x, y), \quad A_\sigma(x) \in GL_n(\mathcal{H}_K(X)). \quad (5.15)$$

Proof. By the properties of the cocycles one has $Y(\sigma(x), y) \cdot Y(x, y)^{-1} = Y(\sigma(x), y) \cdot Y(y, x) = Y(\sigma(x), x)$. Now the matrix

$$A_\sigma(x) := Y(\sigma(x), x) \quad (5.16)$$

converges on X by proposition 5.6. Since $Y(x,y)$ is invertible so does $A_\sigma(x)$. In particular $A_\sigma(x)^{-1} = Y(x, \sigma(x)) = \sum_n G_n(\sigma(x))(-\delta_\sigma(x))^n$. By construction, $Y(x,y)$ satisfies equation (5.15). \square

5.3.2 *Σ -compatible differential equations and Taylor admissible Σ -modules.* Let $\Sigma \subseteq \mathfrak{S}_X$ be a family of infinitesimal operators of $\mathcal{H}_K(X)$. We denote by

$$d - \text{Mod}(\mathcal{H}_K(X))^{\text{adm}(\Sigma)}, \quad (\text{resp. } \Sigma - \text{Mod}(\mathcal{H}_K(X))^{\text{adm}}) \quad (5.17)$$

the full subcategory of $d - \text{Mod}(\mathcal{H}_K(X))$ (resp. $\Sigma - \text{Mod}(\mathcal{H}_K(X))$) whose objects admit a Σ -compatible analytic cocycle as solution (cf. Corollary 5.8). We call the objects of $d - \text{Mod}(\mathcal{H}_K(X))^{\text{adm}(\Sigma)}$ Σ -admissible differential equations, and call Taylor admissible Σ -modules the objects in $\sigma - \text{Mod}(\mathcal{H}_K(X))^{\text{adm}}$.

By the properties of analytic cocycles one verifies that $d - \text{Mod}(\mathcal{H}_K(X))^{\text{adm}(\Sigma)}$, $\Sigma - \text{Mod}(\mathcal{H}_K(X))^{\text{adm}}$ are \otimes -categories (cf. lemma 3.11).

5.3.3 *Definition of the deformation functor.* Let X be an affinoid. Let $\Sigma \subseteq \mathfrak{S}_X$ be a family of infinitesimal automorphisms.

We consider $(\mathcal{H}_K(X), \Sigma)$ as a generalized Σ -algebra (where $i_\sigma = \text{Id}_{\mathcal{H}_K(X)}$, for all $\sigma \in \Sigma$). We consider also the Σ_{Strat} -algebra structure on $\mathcal{H}_K(X)$ introduced in section 4.3.1. The category $d - \text{Mod}(\mathcal{H}_K(X))^{\text{adm}(\Sigma)}$ is then the fully faithful sub-category of $d - \text{Mod}(\mathcal{H}_K(X))$ formed by differential equations whose stratifications lies over $\mathcal{A}_K(\Sigma)$ (i.e. whose Taylor solution belongs to $GL_n(\mathcal{A}_K(\Sigma))$).

We then observe that $(\mathcal{H}_K(X), \Sigma)$ admits a natural structure of Σ_{Strat} -algebra over $(\mathcal{H}_K(X), \Sigma_{\text{Strat}})$ induced by the family of morphisms $\{\Delta_\sigma : \mathcal{A}_K(\Sigma) \mapsto \mathcal{H}_K(X)\}_{\sigma \in \Sigma}$. Namely, with the notation of section 4.2.2, for all $\sigma \in \Sigma$, one has $j := \text{Id}_{\mathcal{H}_K(X)}$, and $\Delta_\sigma(f(x, y)) := f(\sigma(x), x)$ (cf. Proposition 5.6) as exposed by the following diagrams:

$$\begin{array}{ccc} \mathcal{H}_K(X) & \xrightarrow{\sigma} & \mathcal{H}_K(X) \\ \text{Id} \uparrow & & \uparrow \Delta_\sigma \\ \mathcal{H}_K(X) & \xrightarrow{p_1} & \mathcal{A}_K(\mathcal{T}(X, R)) \end{array} \quad ; \quad \begin{array}{ccc} \mathcal{H}_K(X) & \xrightarrow{\text{Id}} & \mathcal{H}_K(X) \\ \text{Id} \uparrow & & \uparrow \Delta_\sigma \\ \mathcal{H}_K(X) & \xrightarrow{p_2} & \mathcal{A}_K(\mathcal{T}(X, R)) \end{array} \quad (5.18)$$

Here the projection $\mathcal{P} : \Sigma \rightarrow \Sigma_{\text{Strat}} = \{p_1\}$ is the evident map sending every σ into p_1 . We define then the functor

$$\text{Def}_\Sigma : d - \text{Mod}(\mathcal{H}_K(X)) \rightarrow \Sigma - \text{Mod}(\mathcal{H}_K(X)) \quad (5.19)$$

as the composite

$$\text{Def}_\Sigma := \text{Id}_\Delta^* \circ S, \quad (5.20)$$

where $S : d - \text{Mod}(\mathcal{H}_K(X)) \xrightarrow{\sim} \text{Strat}(\mathcal{H}_K(X))$ is the equivalence defined in section 3.4, and Id_Δ^* is the pull back functor defined in section 4.2.2.

5.3.4 We notice that

- i) The underling $\mathcal{H}_K(X)$ -module of $\text{Def}_\Sigma(M)$ is equal to M ;
- ii) Def_Σ is the identity on the morphisms: $\text{Def}_\Sigma(\alpha) = \alpha$, for all $\alpha : (M, \nabla^M) \rightarrow (N, \nabla^N)$. In particular Def_Σ is a faithful functor (see also Theorem 5.18 for the fully faithfulness).
- iii) One has $\text{Def}(M, \nabla^M) = \{ \sigma^M : \sigma^* M \xrightarrow{\sim} i_\sigma^* M \}_{\sigma \in \Sigma}$, with $\sigma^M = \Delta_\sigma^*(\chi_M)$ (cf. (3.12)), for all $\sigma \in \Sigma$. In particular if M is represented, in the basis $\mathbf{e} \subset M$, by the equation $Y' = G(z)Y$, and if $Y(x, y)$ denotes the regular cocycle in this basis, then the matrix of σ^M in the same basis $\mathbf{e} \subset M$ is given by (cf. section 5.3.1)

$$A_\sigma(x) := Y(\sigma(x), x) \in GL_n(\mathcal{H}_K(X)), \quad \text{for all } \sigma \in \Sigma. \quad (5.21)$$

5.4 Fully faithfulness and non degeneracy

In this section we discuss the fully faithfulness of the Deformation functor. For this we need the notion of *non degenerate (families of infinitesimal) automorphisms*.

5.4.1 Non degenerate families of automorphisms of $\mathcal{H}_K(X)$.

DEFINITION 5.9. Let Ω/K be an extension of complete valued fields. Let σ be an infinitesimal automorphism of X , and let $y \in X(\Omega)$ be a Ω -rational point. We set

$$\rho_{y,X}(\sigma) := \inf(\rho \text{ such that } \rho \leq \rho_{y,X}, \text{ and } D^-(y, \rho) \text{ is invariant under } \sigma). \quad (5.22)$$

LEMMA 5.10. *The following holds:*

- i) $\rho_{y,X}(\sigma) < \rho_{y,X}$.
- ii) $\rho_{y,X}(\sigma) = \inf(\rho < \rho_{y,X} \text{ such that } |\delta_\sigma|_{y,\rho} < \rho)$.
- iii) Every disk $D^-(y, R)$ with $\rho_{y,X}(\sigma) < R \leq \rho_{y,X}$ is invariant under σ .
- iv) If y is fixed by σ , then $\rho_{y,X}(\sigma) = 0$.
- v) If, for all Ω/K , σ has no Ω -rational fixed points in $D^-(y, \rho_{y,X})$ (i.e. if δ_σ has no zeros in $D^-(y, \rho_{y,X})$, cf. Section 2.2.1), then $\rho_{y,X}(\sigma) = |\delta_\sigma(y)|_\Omega$.

Proof. Let $R \leq \rho_{y,X}$. Since $\sigma(x) - y = \delta_\sigma(x) + x - y$, one sees that $D^-(y, R)$ is invariant under σ if and only if $|\delta_\sigma(x)| < R$ for all $|x - y| < R$. In other words $D^-(y, R)$ is invariant if and only if $\sup_{|x-y|<R} |\delta_\sigma(x)| < R$. By section 2.2.1 this means $|\delta_\sigma|_{y,R} < R$. \square

DEFINITION 5.11. Let $\Sigma \subseteq \mathfrak{S}_X$ be a family of infinitesimal automorphisms of X . We will say that Σ is a *non degenerate family of automorphisms of $\mathcal{H}_K(X)$* , if there exists a valued field extension Ω/K , and an Ω -rational point $y_0 \in X(\Omega)$ such that, for all R satisfying $\rho_{y_0,X}(\sigma) < R \leq \rho_{y_0,X}$, for all $\sigma \in \Sigma$, one has

$$\mathcal{A}_\Omega(y_0, R)^\Sigma = \Omega, \quad (5.23)$$

where $\mathcal{A}_\Omega(y_0, R)^\Sigma$ denotes as usual the sub-ring formed by the functions fixed by every element of Σ . We will say that y_0 is a *base point* for Σ .

REMARK 5.12. i) If Σ is non degenerate, then $\mathcal{H}_K(X)^\Sigma = K$. Indeed the morphism $\mathcal{H}_K(X) \rightarrow \mathcal{A}_\Omega(y_0, R)$ is injective, hence $\mathcal{H}_K(X)^\Sigma = \mathcal{A}_\Omega(y_0, R)^\Sigma \cap \mathcal{H}_K(X) = \Omega \cap \mathcal{H}_K(X) = K$.

ii) If Σ is non degenerate and if $\Sigma \subseteq \Sigma'$, then Σ' is non degenerate.

LEMMA 5.13 (Criterion of non degeneracy). *Let $\Sigma \subseteq \mathfrak{S}_X$ be a family of infinitesimal automorphisms of X . Assume that there exists a Berkovich point $|\cdot|_* \in \mathcal{M}(X)$ such that $0 \in \mathbb{R}$ belongs to the closure (in $\mathbb{R}_{\geq 0}$) of the set $\{|\delta_{\sigma^n}|_*\}_{\sigma \in \Sigma, n \geq 1} - \{0\}$. Then the family Σ is non degenerate.*

Proof. Let $t_* \in X(\Omega)$ be a Dwork generic point for $|\cdot|_*$, and let R be such that $\rho_{t_*,X}(\sigma) < R \leq \rho_{t_*,X}$, for all $\sigma \in \Sigma$. If $f \in \mathcal{A}_\Omega(t_*, R)$ is fixed by Σ , then $|f(t_*)| = |f(\sigma^n(t_*))|$ for all $n \geq 0$ and all $\sigma \in \Sigma$. By assumption there exists sequences $\{n_m\}_m$ and $\{\sigma_m\}_m$ such that $\sigma_m^{n_m}(t_*)$ tends to t_* and $\sigma_m^{n_m}(t_*) \neq \sigma_{m'}^{n_{m'}}(t_*)$, for all $m \neq m'$. So the function $g(T) := f(T) - f(t_*)$ has infinitely many zeros tending to t_* . So $g(T) = 0$, and f is constant. \square

LEMMA 5.14 (Uniqueness of the cocycle). *Let $\Sigma \subseteq \mathfrak{S}_X$ be a non degenerate family of automorphisms of X . Let $\{\sigma(Y) = A_\sigma(T) \cdot Y\}_{\sigma \in \Sigma}$ be a family of σ -difference equations. If there exists a Σ -compatible analytic cocycle $Y(x, y)$ which is simultaneously solution of every equation of this family, then this cocycle is unique.*

Proof. Let $y_0 \in X(\Omega)$ be a base point for Σ . Assume that $Y_1(x, y), Y_2(x, y)$ are both solution of the same family equation $\{\sigma(Y) = A_\sigma Y\}_{\sigma \in \Sigma}$. For all $\sigma \in \Sigma$ one has $Y_1(\sigma(x), y)^{-1} = Y_1(x, y)^{-1} A_\sigma^{-1}$,

hence $F(\sigma(x), y) = F(x, y)$, where $F(x, y) = Y_1(x, y)^{-1}Y_2(x, y)$. Since $\mathcal{A}_\Omega(y_0, R)^\Sigma = \Omega$, the function $F(x, y_0) \in GL_n(\mathcal{A}_\Omega(y_0, R))$ belongs to $GL_n(\Omega)$, and moreover for $x = y_0$ one has $F(y_0, y_0) = \text{Id}$. So $F(x, y_0) = \text{Id}$ i.e. $Y_1(x, y_0) = Y_2(x, y_0)$. By Lemma 3.8 one has $Y_1(x, y) = Y_2(x, y)$. \square

LEMMA 5.15. *The analogous of Lemma 3.9 holds for Σ -modules (providing that $\Sigma \subseteq \mathfrak{S}_X$ and that the analytic cocycle $Y(x, y)$ is Σ -compatible)* \square

5.4.2 Fully faithfulness of the deformation functor.

DEFINITION 5.16. We define

$$(d, \Sigma) - \text{Mod}(\mathcal{H}_K(X))^{\text{adm}} \quad (5.24)$$

as the category of triplets $(M, \nabla^M, \{\sigma^M\}_{\sigma \in \Sigma})$ in which

- i) (M, ∇^M) and $(M, \{\sigma^M\}_{\sigma \in \Sigma})$ belong to $d - \text{Mod}(\mathcal{H}_K(X))^{\text{adm}(\Sigma)}$ and $\Sigma - \text{Mod}(\mathcal{H}_K(X))^{\text{adm}}$ respectively,
- ii) (M, ∇^M) and $(M, \{\sigma^M\}_{\sigma \in \Sigma})$ admit, in a given basis (and hence in every basis), the same Σ -compatible analytic cocycle $Y(x, y)$ as solution.

In other words, for all $\sigma \in \Sigma$, the automorphism σ^M is deduced by σ -Deformation from ∇^M . Morphisms $\alpha : (M, \nabla^M, \{\sigma^M\}_{\sigma \in \Sigma}) \rightarrow (N, \nabla^N, \{\sigma^N\}_{\sigma \in \Sigma})$ are $\mathcal{H}_K(X)$ -linear maps commuting with the actions of ∇ and σ , for all $\sigma \in \Sigma$.

Consider now the following functors:

$$d - \text{Mod}(\mathcal{H}_K(X))^{\text{adm}(\Sigma)} \xleftarrow[\sim]{\text{Forget } \Sigma} (d, \Sigma) - \text{Mod}(\mathcal{H}_K(X))^{\text{adm}(\Sigma)} \xrightarrow{\text{Forget } \nabla} \Sigma - \text{Mod}(\mathcal{H}_K(X))^{\text{adm}}, \quad (5.25)$$

The above sections prove that the forgetful functor $\text{Forget } \Sigma$ is an equivalence. We are interested to the functor $\text{Forget } \nabla$. For this assume that Σ is non degenerate, and let $y_0 \in X(\Omega)$ be a base point of Σ . Let $R > 0$ be a real number satisfying $\rho_{y_0}(\sigma) < R \leq \min(\text{Rad}(Y_M(x, y), y_0), \text{Rad}(Y_N(x, y), y_0))$, where Y_M and Y_N are the cocycles attached to M and N respectively. Denote by $M_{y_0} := M \otimes_{\mathcal{H}_K(X)} \mathcal{A}_K(y_0, R)$, $N_{y_0} := \otimes_{\mathcal{H}_K(X)} \mathcal{A}_K(y_0, R)$. We have then the following inclusions

$$\text{Hom}_{\mathcal{H}_K(X)}^\nabla(M, N) = \text{Hom}_{\mathcal{A}_\Omega(y_0, R)}^\nabla(M_{y_0}, N_{y_0}) \cap \text{Hom}_{\mathcal{H}_K(X)}(M, N), \quad (5.26)$$

$$\text{Hom}_{\mathcal{H}_K(X)}^\Sigma(M, N) = \text{Hom}_{\mathcal{A}_\Omega(y_0, R)}^\Sigma(M_{y_0}, N_{y_0}) \cap \text{Hom}_{\mathcal{H}_K(X)}(M, N). \quad (5.27)$$

To prove the fully faithfulness it is hence enough to prove that the inclusion

$$\text{Hom}_{\mathcal{A}_\Omega(y_0, R)}^\nabla(M_{y_0}, N_{y_0}) \subseteq \text{Hom}_{\mathcal{A}_\Omega(y_0, R)}^\Sigma(M_{y_0}, N_{y_0}) \quad (5.28)$$

is an equality.

LEMMA 5.17. *Assume that $(M, \nabla^M) \in d - \text{Mod}(\mathcal{H}_K(X))^{\text{adm}(\Sigma)}$. Let $Y(x, y)$ be the cocycle attached to (M, ∇^M) in the basis \mathbf{e} . For all $\sigma \in \Sigma$, let σ^M be the σ -semilinear operator on M deduced by Deformation from ∇^M . Then, for all complete valued field Ω/K , and all $y_0 \in X(\Omega)$, and all $R > 0$ satisfying $\rho_{y, X}(\sigma) < R \leq \text{Rad}(Y(x, y), y_0)$ for all $\sigma \in \Sigma$, there exists a basis of M_{y_0} in which all the operators $\nabla^{M_{y_0}}$ and $\{\sigma^{M_{y_0}}\}_{\sigma \in \Sigma}$ are simultaneously trivial (i.e. equal to direct sum of the unit object).*

Proof. By construction $Y(x, y)$ is also the analytic cocycle attached to (M, σ^M) . Since $Y(x, y)$ is assumed to be Σ -compatible, then $D^-(y_0, R)$ is invariant under the action of every $\sigma \in \Sigma$ (cf. Lemma 5.10). The matrix $Y(x, y_0) \in GL_n(\mathcal{A}_K(y_0, R))$ is a simultaneous solution of $(M_{y_0}, \nabla^{M_{y_0}})$ and $(M_{y_0}, \sigma^{M_{y_0}})$ with values in $\mathcal{A}_K(y_0, R)$, in the basis $\mathbf{e} \otimes 1 \subset M_{y_0}$. Hence in the basis $Y(x, y_0)^{-1} \cdot (\mathbf{e} \otimes 1)$ the new solution of $(M_{y_0}, \nabla^{M_{y_0}})$ and $(M_{y_0}, \sigma^{M_{y_0}})$ is the identity (cf. Lemma 3.9). They are hence all trivial in this basis. \square

The above inclusion (5.28) become hence

$$\Omega^{nm} = \text{Hom}_{\mathcal{A}_{\Omega}(y_0, R)}^{\nabla}(\mathbf{M}_{y_0}, \mathbf{N}_{y_0}) \subseteq \text{Hom}_{\mathcal{A}_{\Omega}(y_0, R)}^{\Sigma}(\mathbf{M}_{y_0}, \mathbf{N}_{y_0}) = (\mathcal{A}_{\Omega}(y_0, R)^{\Sigma})^{nm}. \quad (5.29)$$

Now since the family Σ is non degenerate, one has $\mathcal{A}_{\Omega}(y_0, R)^{\Sigma} = \Omega$ and equality holds. This proves that the functor $\text{Forget } \nabla$ is an equivalence in this case. We summarize the previous facts in the following

THEOREM 5.18. *Assume that Σ is a family of infinitesimal automorphisms of $\mathcal{H}_K(X)$. Then:*

i) *The functor*

$$\text{Forget } \Sigma : (d, \Sigma) - \text{Mod}(\mathcal{H}_K(X))^{\text{adm}} \xrightarrow{\sim} d - \text{Mod}(\mathcal{H}_K(X))^{\text{adm}(\Sigma)} \quad (5.30)$$

is an equivalence, in particular there exists a deformation functor

$$\text{Def}_{\Sigma} := (\text{Forget } \nabla) \circ (\text{Forget } \Sigma)^{-1} : d - \text{Mod}(\mathcal{H}_K(X))^{\text{adm}(\Sigma)} \longrightarrow \Sigma - \text{Mod}(\mathcal{H}_K(X))^{\text{adm}}, \quad (5.31)$$

which is faithful.

ii) *If Σ is non degenerate, then the functor*

$$\text{Forget } \nabla : (d, \Sigma) - \text{Mod}(\mathcal{H}_K(X))^{\text{adm}} \xrightarrow{\sim} \Sigma - \text{Mod}(\mathcal{H}_K(X))^{\text{adm}} \quad (5.32)$$

is an equivalence. In this case the Σ -Deformation functor is an equivalence too. \square

5.5 The case of $\mathcal{H}_K^{\dagger}(X)$

We preserve the notation of Section 1.2. An infinitesimal automorphism of $\mathcal{H}_K(X_{\varepsilon})$ extends uniquely to an infinitesimal automorphism of every $\mathcal{H}_K(X_{\varepsilon'})$, for all $0 \leq \varepsilon' < \varepsilon$. In particular σ defines an automorphism of $\mathcal{H}_K(X)$ and $\mathcal{H}_K^{\dagger}(X)$, whose action on $\mathcal{H}_K(X)$ is infinitesimal. Reciprocally if an infinitesimal automorphism σ of $\mathcal{H}_K(X)$ stabilizes $\mathcal{H}_K(X_{\varepsilon}) \subset \mathcal{H}_K(X)$, then there exists $0 < \varepsilon' < \varepsilon$ such that σ acts infinitesimally on $\mathcal{H}_K(X_{\varepsilon'})$. This results easily by the continuity properties of the function $|\cdot|_* \mapsto |\delta_{\sigma}|_*$ (cf. Propositions 2.13 and 5.3).

DEFINITION 5.19. An automorphism of $\mathcal{H}_K^{\dagger}(X)$ is called infinitesimal if it comes from an automorphism of $\mathcal{H}_K(X_{\varepsilon})$ for some $\varepsilon > 0$ as above, and if it is infinitesimal on $\mathcal{H}_K(X)$.

If Σ is a family of infinitesimal automorphisms of $\mathcal{H}_K^{\dagger}(X)$, we denote by Σ_{ε} the sub-family of Σ acting *infinitesimally* on $\mathcal{H}_K(X_{\varepsilon})$. For $\varepsilon > \varepsilon' > 0$ one has $\Sigma_{\varepsilon} \subset \Sigma_{\varepsilon'}$, and $\Sigma = \cup_{\varepsilon} \Sigma_{\varepsilon}$.

The family Σ of automorphisms of \mathcal{H}_K^{\dagger} will be called *non degenerate* if for all $\varepsilon > 0$ there exists $0 < \varepsilon' < \varepsilon$ such that $\Sigma_{\varepsilon'}$ is non degenerate on $\mathcal{H}_K(X_{\varepsilon'})$.

An analytic cocycle over $\mathcal{H}_K^{\dagger}(X)$ is defined as a cocycle over X_{ε} , for some $\varepsilon > 0$.

LEMMA 5.20. *An analytic cocycle over X_{ε} is Σ -compatible over X if and only if it is Σ -compatible over $X_{\varepsilon'}$, for some $0 < \varepsilon' < \varepsilon$.*

Proof. This results by Propositions 2.13, 5.3, and 5.5. Indeed, by the continuity of $\text{Rad}(Y(x, y), -)$ and of $\delta_{\sigma}(-)$, and by their log-properties the condition “ $\text{Rad}(Y(x, y), |\cdot|_*) < \delta_{\sigma}(|\cdot|_*)$, for all $|\cdot|_* \in \mathcal{M}(X)$ ”, extends to some X_{ε} . \square

REMARK 5.21. Notice that in general there is no $\varepsilon > 0$ such that $Y(x, y)$ is σ -compatible for all $\sigma \in \Sigma$. Indeed ε will depends on the single σ .

One defines the categories $d - \text{Mod}(\mathcal{H}_K^{\dagger}(X))^{\text{adm}(\Sigma)}$, $(d, \Sigma) - \text{Mod}(\mathcal{H}_K^{\dagger}(X))^{\text{adm}}$, $\Sigma - \text{Mod}(\mathcal{H}_K^{\dagger}(X))^{\text{adm}}$ as the inductive limit of the same categories over X_{ε} . For a more detailed description of these objects the reader can imitate the description given in section 6.1 in the case of the Robba ring \mathcal{R}_K . One has the following

THEOREM 5.22. *Theorem 5.18 holds without changes replacing $\mathcal{H}_K(X)$ with $\mathcal{H}_K^\dagger(X)$.* \square

5.6 The case of an open annulus

To simplify the notations we assume the center of the annulus to be equal to 0. Let $I \subseteq \mathbb{R}_{\geq 0}$ be an interval. We recall that if J is a compact interval, then $\mathcal{A}_K(J) = \mathcal{H}_K(\mathcal{C}(J))$. Moreover for all I one has

$$\mathcal{A}_K(I) = \varprojlim_{\substack{J \subseteq I \\ J \text{ compact}}} \mathcal{H}_K(\mathcal{C}(J)) . \quad (5.33)$$

In this section we extend the previous equivalence to the case of a not necessarily closed annulus.

We define analytic cocycles on $\mathcal{C}(I)$ as two variable analytic functions whose restriction to $\mathcal{C}(J) \times \mathcal{C}(J)$ is an analytic cocycle for all compact $J \subset I$. The functions $\rho_{|\cdot|_*, \mathcal{C}(J)}$, $\text{Rad}(Y(x, y), |\cdot|_*)$, $\delta_\sigma(|\cdot|_*)$, defined for all $J \subseteq I$, glue to give functions on $\mathcal{C}(I)$. Notice that for all $\rho \in I$ one has:

$$\rho_{|\cdot|_*, \mathcal{C}(I)} = \begin{cases} \sup(I) & \text{if } 0 \in I \text{ (disk)} \\ \rho & \text{if } 0 \notin I \text{ (annulus)} . \end{cases} \quad (5.34)$$

If J is compact, the family $\mathcal{S} = \{|\cdot|_\rho\}_{\rho \in J}$, is an exhaustive family for the Berkovich analytic space $\mathcal{M}(\mathcal{C}(J))$. Hence infinitesimality condition of Lemma 5.3 becomes

$$\delta_\sigma(|\cdot|_\rho) < \begin{cases} \sup(I) & \text{if } 0 \in J \\ \rho & \text{if } 0 \notin J , \end{cases} \quad \text{for all } \rho \in J . \quad (5.35)$$

If $0 \notin I$, this condition implies that infinitesimal automorphisms σ of $\mathcal{A}_K(J)$, have the property that, for all compact $J' \subset J$, σ acts on $\mathcal{A}_K(J')$ and this action is infinitesimal.

If $I \subseteq \mathbb{R}_{> 0}$ is now an arbitrary interval, we say that $\sigma : \mathcal{A}_K(I) \rightarrow \mathcal{A}_K(I)$ is *infinitesimal* if it acts infinitesimally on every compact sub-annulus $\mathcal{A}_K(J) \subseteq \mathcal{A}_K(I)$. We say moreover that a family Σ is *non degenerate* over $\mathcal{A}_K(I)$ if its restriction to some compact J is non degenerate. The above discussion proves the following

LEMMA 5.23. *Assume that $0 \notin I$. Let $\sigma : \mathcal{A}_K(I) \xrightarrow{\sim} \mathcal{A}_K(I)$ be a K -linear continuous automorphism of the form $\sigma(f(T)) = f(\sigma(T))$. Then σ is infinitesimal on $\mathcal{A}_K(I)$ if and only if it verifies $\delta_\sigma(|\cdot|_\rho) < \rho$, for all $\rho \in I$.* \square

We say that a cocycle $Y(x, y)$ is Σ -compatible over $\mathcal{C}(I)$ if and only if its restriction to every $\mathcal{C}(J)$, with $J \subseteq I$ compact, is Σ -compatible.

5.6.1 *The categories $d - \text{Mod}(\mathcal{A}_K(I))^{\text{adm}(\Sigma)}$ and $\Sigma - \text{Mod}(\mathcal{A}_K(I))^{\text{adm}}$.* We define

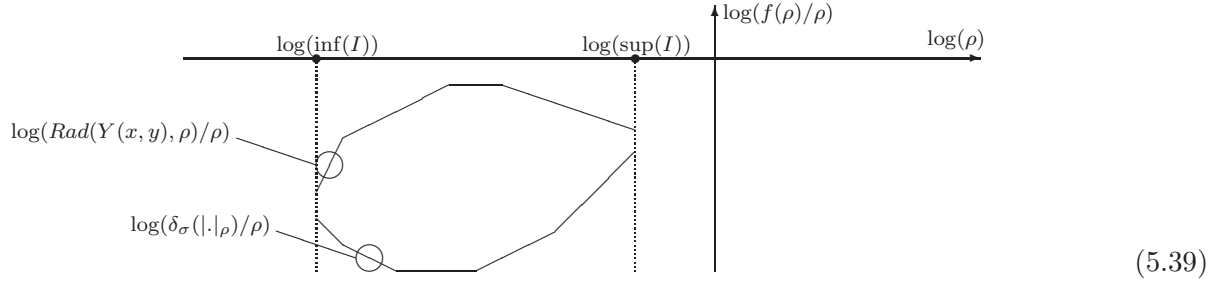
$$d - \text{Mod}(\mathcal{A}_K(I))^{\text{adm}(\Sigma)} := \varprojlim_{\substack{J \subseteq I \\ J \text{ compact}}} d - \text{Mod}(\mathcal{H}_K(\mathcal{C}(J)))^{\text{adm}(\Sigma)} , \quad (5.36)$$

$$\Sigma - \text{Mod}(\mathcal{A}_K(I))^{\text{adm}} = \varprojlim_{\substack{J \subseteq I \\ J \text{ compact}}} \Sigma - \text{Mod}(\mathcal{H}_K(\mathcal{C}(J)))^{\text{adm}} , \quad (5.37)$$

$$(d, \Sigma) - \text{Mod}(\mathcal{A}_K(I))^{\text{adm}} = \varprojlim_{\substack{J \subseteq I \\ J \text{ compact}}} (d, \Sigma) - \text{Mod}(\mathcal{H}_K(\mathcal{C}(J)))^{\text{adm}} . \quad (5.38)$$

In other words $d - \text{Mod}(\mathcal{A}_K(I))^{\text{adm}(\Sigma)}$ is the full subcategory of $d - \text{Mod}(\mathcal{A}_K(I))$ whose objects (M, ∇^M) satisfy the property that for all compact $J \subseteq I$ the restriction of (M, ∇^M) to J belongs to $d - \text{Mod}(\mathcal{H}_K(\mathcal{C}(J)))^{\text{adm}(\Sigma)}$. The same definition is given for $\Sigma - \text{Mod}(\mathcal{A}_K(I))^{\text{adm}}$, and $(d, \Sigma) - \text{Mod}(\mathcal{A}_K(I))^{\text{adm}}$.

LEMMA 5.24. Let $Y(x, y)$ be a cocycle on $\mathcal{C}(I)$. Let σ be an infinitesimal automorphism of $\mathcal{A}_K(I)$. Then $Y(x, y)$ is σ -compatible if and only if $\delta_\sigma(|\cdot|_\rho) < \text{Rad}(Y(x, y), \rho)$, for all $\rho \in I$.



Proof. See Proposition 5.5. \square

REMARK 5.25. Since $\rho \mapsto \text{Rad}(Y(x, y), \rho)$ is log-concave, and $\rho \mapsto \delta_\sigma(|\cdot|_\rho)$ is log-convex, then it is enough to check the condition $|\delta_\sigma|_\rho < \text{Rad}(Y(x, y), \rho)$ for ρ close to $\sup(I)$ and $\inf(I)$.

REMARK 5.26. Notice that, if I is not compact, the uniform neighborhood \mathcal{U} on which $Y(x, y)$ converges, does not contain necessarily a subset of the form $\mathcal{T}(\mathcal{C}(I), R) = \{ (x, y) \in \mathcal{C}(I) \times \mathcal{C}(I) \mid |x - y| < R \}$. This is however true for the restriction of $Y(x, y)$ to every compact $J \subset I$.

We then have the following:

THEOREM 5.27. Theorem 5.18 holds without changes replacing $\mathcal{H}_K(X)$ with $\mathcal{A}_K(I)$. \square

6. σ -modules and Differential equations over the Robba ring

By definition one has $\mathcal{R}_K = \bigcup_{\varepsilon > 0} \mathcal{A}_K([1 - \varepsilon, 1])$. To simplify the notations we set

$$I_\varepsilon :=]1 - \varepsilon, 1[\quad (6.1)$$

Let $\sigma : \mathcal{A}_K(I_\varepsilon) \xrightarrow{\sim} \mathcal{A}_K(I_\varepsilon)$ be an infinitesimal automorphism. Since σ acts (infinitesimally) on every sub annulus $I_{\varepsilon'}$, with $\varepsilon' < \varepsilon$, then it acts on \mathcal{R}_K . We say that an automorphism $\sigma : \mathcal{R}_K \xrightarrow{\sim} \mathcal{R}_K$ is *infinitesimal* if there exists an $\varepsilon > 0$ and an infinitesimal automorphism σ_ε of $\mathcal{A}_K([1 - \varepsilon, 1])$, such that σ is induced by σ_ε , as above.

Let $\Sigma \subseteq \mathfrak{S}_X$ be a family of infinitesimal automorphisms of \mathcal{R}_K . Every element of Σ lives over an annulus $\mathcal{A}_K(I_\varepsilon)$, we denote by Σ_ε the set of automorphisms in Σ which live over I_ε and acts infinitesimally over $\mathcal{A}_K(I_\varepsilon)$. For $\varepsilon > \varepsilon'$ one has $\Sigma_\varepsilon \subset \Sigma_{\varepsilon'}$, and $\Sigma = \bigcup_\varepsilon \Sigma_\varepsilon$.

We say that the family Σ is *non degenerate* over \mathcal{R}_K if for all $\varepsilon > 0$ there exists $0 < \varepsilon' < \varepsilon$ such that $\Sigma_{\varepsilon'}$ is non degenerate on $\mathcal{A}_K(I_{\varepsilon'})$.

An analytic cocycle over \mathcal{R}_K is a germ of analytic cocycle, i.e. an analytic cocycle over some unspecified I_ε .

6.1 Deformation equivalence over the Robba ring

Let $\Sigma = \{\sigma_\lambda\}_{\lambda \in \Lambda}$ be a family of infinitesimal automorphisms of \mathcal{R}_K . The categories

$$d - \text{Mod}(\mathcal{R}_K)^{\text{adm}(\Sigma)}, \quad \Sigma - \text{Mod}(\mathcal{R}_K)^{\text{adm}}, \quad (d, \Sigma) - \text{Mod}(\mathcal{R}_K)^{\text{adm}} \quad (6.2)$$

are defined as the inductive limit of the categories $d - \text{Mod}(\mathcal{A}_K(I_\varepsilon))^{\text{adm}(\Sigma_\varepsilon)}$, $\Sigma_\varepsilon - \text{Mod}(\mathcal{A}_K(I_\varepsilon))^{\text{adm}}$, and $(d, \Sigma_\varepsilon) - \text{Mod}(\mathcal{A}_K(I_\varepsilon))^{\text{adm}}$ respectively. In other words:

- i) An \mathcal{R}_K -differential module belongs to $d - \text{Mod}(\mathcal{R}_K)^{\text{adm}(\Sigma)}$ if there exists $\varepsilon' > 0$ such that for all $0 < \varepsilon < \varepsilon'$, M comes by scalar extension from a Σ_ε -compatible differential module M_ε over $\mathcal{A}_K(I_\varepsilon)$.

- ii) A Σ -module $(M, \{\sigma^M\}_{\sigma \in \Sigma})$ belongs to $\Sigma - \text{Mod}(\mathcal{R}_K)^{\text{adm}}$ if there exists a $\varepsilon' > 0$ such that, for all $0 < \varepsilon < \varepsilon'$, its restriction $(M, \{\sigma^M\}_{\sigma \in \Sigma_\varepsilon})$ to Σ_ε comes by scalar extension from a Taylor admissible Σ_ε -module over $\mathcal{A}_K(I_\varepsilon)$.
- iii) A (d, Σ) -module $(M, \nabla^M, \{\sigma_\lambda^M\}_{\lambda \in \Lambda})$ belongs to $(d, \Sigma) - \text{Mod}(\mathcal{R}_K)^{\text{adm}}$ if there exists $\varepsilon' > 0$ such that, for all $0 < \varepsilon < \varepsilon'$, its restriction $(M, \nabla^M, \{\sigma^M\}_{\sigma \in \Sigma_\varepsilon})$ to Σ_ε comes by scalar extension from an admissible (d, Σ_ε) -module over $\mathcal{A}_K(I_\varepsilon)$.

As usual morphisms in $d - \text{Mod}(\mathcal{R}_K)^{\text{adm}(\Sigma)}$ are \mathcal{R}_K -linear maps commuting with the connections, morphisms in $\Sigma - \text{Mod}(\mathcal{R}_K)^{\text{adm}}$ are \mathcal{R}_K -linear maps commuting with the action of Σ , and morphisms in $(d, \Sigma) - \text{Mod}(\mathcal{R}_K)^{\text{adm}}$ are \mathcal{R}_K -linear maps commuting with the action of Σ and with the connections.

THEOREM 6.1. *Theorem 5.18 holds without changes replacing $\mathcal{H}_K(X)$ with \mathcal{R}_K .* \square

6.2 Solvability, slopes and p -adic irregularities

Unless precise mention, from now on until the end of section 6 we will assume that Σ is a non degenerate family of infinitesimal automorphisms of \mathcal{R}_K . In this subsection we assume that K has mixed characteristic $(0, p)$.

We say that a cocycle $Y(x, y)$ over \mathcal{R}_K is *solvable* if one has

$$\lim_{\rho \rightarrow 1^-} \text{Rad}(Y(x, y), \rho) = 1. \quad (6.3)$$

We denote by $d - \text{Mod}(\mathcal{R}_K)^{[1]}$, $d - \text{Mod}(\mathcal{R}_K)^{[1], \text{adm}(\Sigma)}$, $(d, \Sigma) - \text{Mod}(\mathcal{R}_K)^{[1], \text{adm}}$, $\Sigma - \text{Mod}(\mathcal{R}_K)^{[1], \text{adm}}$ the full subcategories of $d - \text{Mod}(\mathcal{R}_K)$, $d - \text{Mod}(\mathcal{R}_K)^{\text{adm}(\Sigma)}$, $(d, \Sigma) - \text{Mod}(\mathcal{R}_K)^{\text{adm}}$, $\Sigma - \text{Mod}(\mathcal{R}_K)^{\text{adm}}$ respectively formed by objects admitting a *solvable* cocycle as solution. One shows (cf. [CM02]) that, if $Y(x, y)$ is solvable over \mathcal{R}_K , then there exists an $\varepsilon > 0$ such that, for all $\rho \in I_\varepsilon$, one has $\text{Ray}(Y(x, y), \rho) = \rho^\beta$, for some $\beta \geq 0$. The number $\beta - 1$ is called the p -adic Slope of the differential equation associated to the cocycle:

$$\text{Slope}(M, \nabla^M) := \beta - 1. \quad (6.4)$$

This definition clearly depends only on $Y(x, y)$ so we may speak of *p -adic Slope of $Y(x, y)$* . This permits to define in an evident way a notion of p -adic Slope of an admissible Σ -module, providing that its Σ -admissible cocycle is solvable. G.Christol and Z.Mebkhout proved that every solvable differential equation admits a so called break decomposition following the slopes. We summarize the main properties in the following theorem.

THEOREM 6.2 ([CM02]). *Let M be a solvable differential module over \mathcal{R}_K . There exists a unique decomposition of M , called break decomposition*

$$M = \bigoplus_{x \in \mathbb{R}_{\geq 0}} M(x), \quad (6.5)$$

satisfying the following properties. Let t_ρ be a Dwork generic point for the norm $|\cdot|_\rho$, then there exists $\varepsilon > 0$ such that

- i) *For all $\rho \in]1 - \varepsilon, 1[$, $M(x)$ is the biggest submodule of M trivialized by $\mathcal{A}_K(t_\rho, \rho^{x+1})$,*
- ii) *For all $\rho \in]1 - \varepsilon, 1[$, and for all $y < x$, $M(x)$ has no solutions in $\mathcal{A}_K(t_\rho, \rho^{y+1})$.*

The number $\text{Irr}(M) := \sum_{x \geq 0} x \cdot \text{rank}_{\mathcal{R}_K}(M(x))$ is called p -adic irregularity of M , and it lies in \mathbb{N} .

Since the slope is defined in term of analytic cocycle, and since the Deformation equivalence preserves the Taylor solutions, it is clear that such a result holds for Taylor admissible Σ -modules. We resume these consideration in the following proposition:

PROPOSITION 6.3. *Let Σ be a non degenerate family of infinitesimal automorphisms of \mathcal{R}_K . There exists a notion of p -adic Slope and p -adic Irregularity of an object M in $\Sigma - \text{Mod}(\mathcal{R}_K)^{[1], \text{adm}}$. The Deformation functor preserves this notion. More precisely let M be a solvable admissible Σ -module over \mathcal{R}_K . There exists a unique decomposition of M , called break decomposition*

$$M = \bigoplus_{x \in \mathbb{R}_{\geq 0}} M(x), \quad (6.6)$$

satisfying the following properties: there exists $\varepsilon_M > 0$ such that for all $\varepsilon > \varepsilon_M$ one has

- i) For all $\rho \in]1 - \varepsilon, 1[$, $M(x)$ is the biggest submodule of M trivialized by $\mathcal{A}_K(t_\rho, \rho^{x+1})$,
- ii) For all $\rho \in]1 - \varepsilon, 1[$, and for all $y < x$, $M(x)$ has no solutions in $\mathcal{A}_K(t_\rho, \rho^{y+1})$.

The number $\text{Irr}(M) := \sum_{x \geq 0} x \cdot \text{rank}_{\mathcal{R}_K}(M(x))$ is called p -adic irregularity of M , and it lies in \mathbb{N} . \square

Let $*$ denotes one of the symbols \leq or $<$, in the sequel we will denote by $d - \text{Mod}(\mathcal{R}_K)^{\text{Sl}*\beta}$, $d - \text{Mod}(\mathcal{R}_K)^{\text{adm}(\Sigma), \text{Sl}*\beta}$, $\Sigma - \text{Mod}(\mathcal{R}_K)^{\text{adm}, \text{Sl}*\beta}$ the full sub categories of $d - \text{Mod}(\mathcal{R}_K)$, $d - \text{Mod}(\mathcal{R}_K)^{\text{adm}(\Sigma)}$, $\Sigma - \text{Mod}(\mathcal{R}_K)^{\text{adm}}$ respectively, whose objects have a p -adic Slope which is $*$ than β .

6.3 Frobenius functor

Let $\phi_K : K \rightarrow K$ be a lifting of the p th power map of the residual field k . We denote by $\phi : \mathcal{R}_K \rightarrow \mathcal{R}_K$ a chosen lifting of the Frobenius $x \mapsto x^p$ of $k((t))$ extending ϕ_K , that is $\phi(f(x)) := f^{\phi_K}(\phi(x))$, where f^{ϕ_K} is the power series deduced from f by applying ϕ_K to its coefficients, and where $\phi(x)$ verifies $|\phi(x) - x^p|_\rho < \rho$ for all ρ sufficiently close to 1. The map ϕ sends $\mathcal{A}_K(]r, 1[)$ into $\mathcal{A}_K(]r^{1/p}, 1[)$.

We define now an endo-functor ϕ^* of the above categories of solvable objects called Frobenius functor. We recall that an analytic cocycle over \mathcal{R}_K is a germ of analytic cocycle $Y(x, y)$ over some $\mathcal{A}_K(]r, 1[)$, $0 < r < 1$. We define the pull-back of $Y(x, y)$ as

$$\phi^*Y(x, y) := Y^{\phi_K}(\phi(x), \phi(y)). \quad (6.7)$$

This is an analytic cocycle over $\mathcal{A}_K(]r^{1/p}, 1[)$ whose radius is given by (cf. [CM02, Prop.7.2])

$$\text{Rad}(\phi^*(Y(x, y)), \rho) = \begin{cases} \text{Rad}(Y(x, y), \rho^p)^{1/p} & \text{if } \text{Rad}(Y(x, y), \rho^p) > (\omega\rho)^p, \\ \text{Rad}(Y(x, y), \rho^p)/(\omega\rho)^{p-1} & \text{if } \text{Rad}(Y(x, y), \rho^p) < (\omega\rho)^p. \end{cases} \quad (6.8)$$

In other words:

$$\text{Rad}(Y(x, y), \rho^p) = \begin{cases} \text{Rad}(\phi^*(Y(x, y)), \rho)^p & \text{if } \text{Rad}(\phi^*(Y(x, y)), \rho) > \omega\rho, \\ \text{Rad}(\phi^*(Y(x, y)), \rho) \cdot |p|\rho^{p-1} & \text{if } \text{Rad}(\phi^*(Y(x, y)), \rho) < \omega\rho. \end{cases} \quad (6.9)$$

If $\text{Rad}(Y(x, y), \rho^p) = (\omega\rho)^p$ we only have

$$\text{Ray}(\phi^*(Y(x, y)), \rho) \geq \max(\text{Rad}(Y(x, y), \rho^p)/(\omega\rho)^{p-1}, \text{Rad}(Y(x, y), \rho^p)^{1/p}).^7 \quad (6.10)$$

However, the value of $\text{Rad}(\phi^*(Y(x, y)), \rho)$ can be deduced by continuity of the Radius in the most part of cases, so that the previous inequality is an equality.

In particular $Y(x, y)$ is solvable if and only if $\phi^*Y(x, y)$ is solvable, and if it is the case the previous relation becomes

$$\text{Rad}(Y(x, y), \rho) = \text{Rad}(\phi^*Y(x, y), \rho), \quad (6.11)$$

for ρ sufficiently close to 1^- . This implies in particular that $Y(x, y)$ is Σ -admissible if and only if $\phi^*Y(x, y)$ is Σ -admissible.

⁷The classical counterexample is the following. Assume that the Frobenius is defined by $\phi(T) = T^p$. One verifies that the rank one analytic cocycle $Y(x, y)$, defined as the Taylor solution of the differential equation $d/dT - p^{-1}T^{-1}$, has radius $\text{Rad}(Y(x, y), \rho) = \omega\rho$, but its pull back $\phi^*(Y(x, y))$ is isomorphic to the trivial cocycle $Y(x, y) = \text{Id}$, so it has radius $\text{Rad}(\text{Id}, \rho) = \rho$, which is strictly larger than the maximum of (6.10).

6.3.1 The Frobenius Functor. The *Frobenius functor* sends a solvable object M associated to the solvable cocycle $Y(x, y)$ over $\mathcal{A}_K(\lfloor r, 1 \rfloor)$ into the solvable object $\phi^*(M)$ associated to the cocycle $\phi^*Y(x, y)$ over $\mathcal{A}_K(\lfloor r^{1/p}, 1 \rfloor)$. The action of ϕ^* on the morphisms is given by $\phi^* : \text{Hom}(M, N) \rightarrow \text{Hom}(\phi^*M, \phi^*N)$, $\phi^*(H(x)) := H^{\phi_K}(\phi(x))$, where $H(x)$ is the matrix of a morphism in given basis (this definition does not depend on the chosen basis).

6.3.2 Frobenius Structure. We say that a solvable object M has a *Frobenius structure* of order $h > 0$ if $(\phi^*)^h M$ is isomorphic to M . In terms of cocycles this is equivalent to the existence of an invertible matrix $H(x) \in GL_n(\mathcal{R}_K)$ satisfying

$$(\phi^*)^h Y(x, y) = H(x)Y(x, y)H(y)^{-1}. \quad (6.12)$$

We denote by $d\text{-Mod}(\mathcal{R}_K)^{(\phi)}$, $d\text{-Mod}(\mathcal{R}_K)^{(\phi), \text{adm}(\Sigma)}$, $(d, \Sigma)\text{-Mod}(\mathcal{R}_K)^{(\phi), \text{adm}}$, $\Sigma\text{-Mod}(\mathcal{R}_K)^{(\phi), \text{adm}}$ the full subcategories of $d\text{-Mod}(\mathcal{R}_K)^{[1]}$, $d\text{-Mod}(\mathcal{R}_K)^{[1], \text{adm}(\Sigma)}$, $(d, \Sigma)\text{-Mod}(\mathcal{R}_K)^{[1], \text{adm}}$, $\Sigma\text{-Mod}(\mathcal{R}_K)^{[1], \text{adm}}$ whose objects admit an unspecified Frobenius structure of some order.

6.4 Special extensions

In this subsection we assume that K is discretely valued, of mixed characteristic $(0, p)$, with perfect residual field k . By a result of Katz (cf. [Kat86]), every finite separable Galois extensions of $k((t))$ corresponds to a so called *special extension* of $k[t, t^{-1}]$. Special extensions of $k[t, t^{-1}]$ are finite étale Galois extensions whose Galois group satisfies some particular conditions (cf. [Kat86]). By the theory of Monsky-Washnitzer (cf. [MW68]), *special extensions* of $k[t, t^{-1}]$ can be lifted (preserving the Galois group) to the so called Special extensions of $\mathcal{O}_K[T, T^{-1}]^\dagger$, where $\mathcal{O}_K[T, T^{-1}]^\dagger$ denotes the Monsky-Washnitzer's weak completion of $\mathcal{O}_K[T, T^{-1}]$. Special extensions of $\mathcal{O}_K[T, T^{-1}]^\dagger$ generates by scalar extension the so called Special extensions of \mathcal{H}_K^\dagger . We call *Special extensions* or equivalently *étale extensions* of \mathcal{R}_K the \mathcal{R}_K -algebras obtained by scalar extension from *Special extensions* of \mathcal{H}_K^\dagger . We need to introduce the following sub-ring of \mathcal{R}_K :

$$\mathcal{E}_K^\dagger := \{ f \in \mathcal{R}_K \mid \lim_{\rho \rightarrow 1^-} |f|_\rho < +\infty \}. \quad (6.13)$$

The ring \mathcal{E}_K^\dagger has two topologies. The first one arises, by restriction, from that of \mathcal{R}_K . For this topology \mathcal{E}_K^\dagger is dense in \mathcal{R}_K . The second topology on \mathcal{E}_K^\dagger is given by the norm $|\cdot|_1$, for which \mathcal{E}_K^\dagger is not complete. Since the valuation of K is discrete, then $(\mathcal{E}_K^\dagger, |\cdot|_1)$ is actually an *Henselian* field with residual field $k((t))$. One has the following inclusions $\mathcal{H}_K^\dagger \subset \mathcal{E}_K^\dagger \subset \mathcal{R}_K$. We introduce \mathcal{E}_K^\dagger because it is a field, and because it is an intermediate object between \mathcal{H}_K^\dagger and \mathcal{R}_K . Special extensions corresponds bijectively to unramified extensions of \mathcal{E}_K^\dagger . The situation is resumed in the following diagram (for more details we refer to [ADV04], [Mat02]):

$$\begin{array}{ccc} \left\{ \begin{array}{l} \text{Special} \\ \text{extensions of } \mathcal{H}_K^\dagger \end{array} \right\} & \xrightarrow[\sim]{-\otimes \mathcal{E}_K^\dagger} & \left\{ \begin{array}{l} \text{Finite unramified} \\ \text{extensions of } \mathcal{E}_K^\dagger \end{array} \right\} & \xrightarrow[\sim]{-\otimes \mathcal{R}_K} & \left\{ \begin{array}{l} \text{Special} \\ \text{ext. of } \mathcal{R}_K \end{array} \right\} \\ \uparrow -\otimes K & \circlearrowleft & \uparrow -\otimes K & & \\ \left\{ \begin{array}{l} \text{Special extensions} \\ \text{of } \mathcal{O}_K[T, T^{-1}]^\dagger \end{array} \right\} & \xrightarrow[\sim]{-\otimes \mathcal{O}_{\mathcal{E}_K^\dagger}} & \left\{ \begin{array}{l} \text{Finite unramified} \\ \text{extensions of } \mathcal{O}_{\mathcal{E}_K^\dagger} \end{array} \right\} & & \\ \downarrow -\otimes k & \circlearrowleft & \downarrow -\otimes k & & \\ \left\{ \begin{array}{l} \text{Special} \\ \text{coverings of } k[t, t^{-1}] \end{array} \right\} & \xrightarrow[\sim]{\text{Pull-back}} & \left\{ \begin{array}{l} \text{Finite separable} \\ \text{extensions of } k((t)) \end{array} \right\} & & \end{array} \quad (6.14)$$

6.5 Katz-Matsuda's Canonical Extension

In this subsection we assume that K is discretely valuated, of mixed characteristic $(0, p)$, with perfect residual field k . We show now how to obtain the analogous of the results of [Mat02] about the canonical extension. We recall briefly the context. We recall that Katz (cf. [Kat86], and [Kat87]) proved that, if F is an arbitrary field of characteristic 0, there exists a full sub-category of $d - \text{Mod}(F[T, T^{-1}])$, formed by the so called Special objects, such that the scalar extension functor $d - \text{Mod}(F[T, T^{-1}]) \rightarrow d - \text{Mod}(F((T)))$ induces an equivalence of categories between the category of Special objects and $d - \text{Mod}(F((T)))$. The section of the scalar extension functor so obtained is called *canonical extension*: $\text{Can} : d - \text{Mod}(F((T))) \rightarrow d - \text{Mod}(F[T, T^{-1}])$.

Following the lines of Section 6.4 we consider \mathcal{H}_K^\dagger and \mathcal{R}_K as completions of lifting in characteristic 0 of $k[t, t^{-1}]$ and $k((t))$ respectively. S.Matsuda (cf. [Mat02]) proved that the scalar extension functor $d - \text{Mod}(\mathcal{H}_K^\dagger)^{(\phi)} \rightarrow d - \text{Mod}(\mathcal{R}_K)^{(\phi)}$ admits a section called *canonical extension*

$$\text{Can} : d - \text{Mod}(\mathcal{R}_K)^{(\phi)} \rightarrow d - \text{Mod}(\mathcal{H}_K^\dagger)^{(\phi)}, \quad (6.15)$$

inducing an equivalence of categories with its essential image.⁸ This means in particular that a differential module with Frobenius structure (M, ∇^M) over \mathcal{R}_K admits a basis in which the matrix of the connection lies in $M_n(\mathcal{H}_K^\dagger)$. In terms of cocycles this is equivalent to ask the existence of a basis in which the associated cocycle $Y(x, y)$ over \mathcal{R}_K admits an extension to an uniform neighborhood of the diagonal of $\mathcal{C}([1 - \varepsilon, 1 + \varepsilon]) \times \mathcal{C}([1 - \varepsilon, 1 + \varepsilon])$, for some $\varepsilon > 0$. This extension is necessarily unique by Lemma 3.8.

One has the following result whose proof is clear:

THEOREM 6.4 (Canonical extension). *Let Σ be a family of infinitesimal operators over \mathcal{H}_K^\dagger . The scalar extension functor $d - \text{Mod}(\mathcal{H}_K^\dagger)^{(\phi)} \rightarrow d - \text{Mod}(\mathcal{R}_K)^{(\phi)}$ commutes with the deformation functors. If moreover the family Σ is non degenerate, since the deformation is an equivalence, there exists a canonical extension functor for the categories of Taylor admissible Σ -modules (resp. (d, Σ) -modules) with Frobenius structure. The canonical extension commutes with the deformation. \square*

$$\begin{array}{ccc} d - \text{Mod}(\mathcal{R}_K)^{(\phi), \text{adm}(\Sigma)} & \xrightarrow{\text{Can}} & d - \text{Mod}(\mathcal{H}_K^\dagger)^{(\phi), \text{adm}(\Sigma)} \\ \uparrow \text{Forget } \Sigma & \circlearrowleft & \uparrow \text{Forget } \Sigma \\ (d, \Sigma) - \text{Mod}(\mathcal{R}_K)^{(\phi), \text{adm}(\Sigma)} & \xrightarrow{\text{Can}} & (d, \Sigma) - \text{Mod}(\mathcal{H}_K^\dagger)^{(\phi), \text{adm}(\Sigma)} \\ \downarrow \text{Forget } d & \circlearrowleft & \downarrow \text{Forget } d \\ \Sigma - \text{Mod}(\mathcal{R}_K)^{(\phi), \text{adm}(\Sigma)} & \xrightarrow{\text{Can}} & \Sigma - \text{Mod}(\mathcal{H}_K^\dagger)^{(\phi), \text{adm}(\Sigma)} \end{array} \quad (6.16)$$

6.6 Étale solutions and the p -adic local monodromy theorem

The so called p -adic local monodromy theorem (cf. [And02], [Ked04], [Meb02]) proves that, up to replace the field K by a finite extension, every differential module with Frobenius structure is *quasi-unipotent*, that is it becomes trivial (i.e. direct sum of the unit object) over an \mathcal{R}_K -differential algebra of the type $\mathcal{R}'_K[\log(T)]$, where \mathcal{R}'_K is finite étale extension of \mathcal{R}_K , and T is the variable of \mathcal{R}_K . This means that the differential equations with Frobenius structure admit a fundamental matrix of solutions with values in some $\mathcal{R}'_K[\log(T)]$. We call *étale solutions* the solutions of these equations with values in some $\mathcal{R}'_K[\log(T)]$.

⁸Actually S.Matsuda proved this result for *quasi-unipotent* differential equations, with Frobenius structure, but by [And02], [Meb02], [Ked04] we know that every differential equation with Frobenius structure is quasi unipotent (cf. Section 6.6).

We shall here prove, by deformation, that the same result holds for the categories of Taylor admissible Σ -modules, and (d, Σ) -modules.

6.6.1 Action of σ on Special extensions. By *infinitesimal automorphism* of \mathcal{E}_K^\dagger we mean the restriction to \mathcal{E}_K^\dagger of an infinitesimal automorphism of \mathcal{R}_K fixing globally \mathcal{E}_K^\dagger .

LEMMA 6.5 (Action of Σ on the logarithm). *Let σ be an infinitesimal automorphism of \mathcal{H}_K^\dagger , (resp $\mathcal{E}_K^\dagger, \mathcal{R}_K$). Then*

$$\log(\sigma(T)) = \log(T) + \log(\sigma(T)/T), \quad \text{with } \log(\sigma(T)/T) \in \mathcal{H}_K^\dagger \quad (\text{resp. } \log(\sigma(T)/T) \in \mathcal{E}_K^\dagger, f \in \mathcal{R}_K). \quad (6.17)$$

Proof. Write $\sigma(T)/T = 1 + \frac{\sigma(T)-T}{T}$. Since σ is infinitesimal $(\sigma(T)-T)/T = \delta_\sigma(T)/T$ has norm < 1 in its domain of definition. Then $\sigma(T)/T$ takes values in the disk $D^-(1, 1)$, the composite $\log(\sigma(T)/T)$ converges in the same domain of $\sigma(T)$. If moreover $\sigma(T) \in \mathcal{E}_K^\dagger$, then $\log(\sigma(T)/T) \in \mathcal{E}_K^\dagger$, because $|\sigma(T)-T|_1 < 1$, and hence the values of $\sigma(T)/T$ do not approach the wedge of the disk $D^-(1, 1)$. \square

LEMMA 6.6 (Action of Σ on Special extensions of \mathcal{H}_K). *Let σ be an infinitesimal automorphism of \mathcal{H}_K^\dagger , let $B/k[t, t^{-1}]$ be a Special extension and let $(\mathcal{H}_K^\dagger)'/\mathcal{H}_K^\dagger$ be the corresponding Special extension. Then σ extends uniquely, up to Galois automorphisms of $\text{Gal}((\mathcal{H}_K^\dagger)'/\mathcal{H}_K^\dagger) \xrightarrow{\sim} \text{Gal}(B/k[t, t^{-1}])$, to a continuous automorphism of $(\mathcal{H}_K^\dagger)'/\mathcal{H}_K^\dagger$ and to $(\mathcal{H}_K^\dagger)'[\log(T)]$. In particular there exists a unique extension of σ inducing the identity on the residual ring B of $(\mathcal{H}_K^\dagger)'$.*

Proof. It follow from the formal properties of the Henselian couples [Ray70]. \square

REMARK 6.7. Infinitesimal automorphisms of \mathcal{H}_K^\dagger extend to \mathcal{E}_K^\dagger and \mathcal{R}_K . But an infinitesimal automorphism of \mathcal{R}_K which stabilizes \mathcal{H}_K^\dagger and \mathcal{E}_K^\dagger may be not infinitesimal over \mathcal{H}_K^\dagger . In particular it may not induces the identity on $k[t, t^{-1}]$ and $k((t))$.

DEFINITION 6.8. Let $\bar{\sigma} : k((t)) \xrightarrow{\sim} k((t))$ be an automorphism of fields. Let $\sigma : \mathcal{E}_K^\dagger \rightarrow \mathcal{E}_K^\dagger$ be a lifting of $\bar{\sigma}$. We say that σ is an *infinitesimal automorphisms* of \mathcal{E}_K^\dagger (resp. *non degenerate*) if it is the restriction of an infinitesimal (resp. non degenerate) automorphism of \mathcal{R}_K .

6.6.2 Let Σ be a family of infinitesimal automorphisms of \mathcal{H}_K^\dagger . In the sequel we will denote again by Σ its unique extension to $\mathcal{E}_K^\dagger, \mathcal{R}_K$, and their Special extensions. By uniqueness Σ commutes with the action of the Galois group of the special extensions. Our aim is to prove the following

THEOREM 6.9 (*p*-adic local monodromy theorem for Taylor admissible Σ -modules). *Assume that the Frobenius ϕ is a Frobenius of \mathcal{H}_K^\dagger . Let Σ be a non degenerate family of infinitesimal automorphisms of \mathcal{H}_K^\dagger . Then every object of $\Sigma - \text{Mod}(\mathcal{R}_K)^{\text{adm}, (\phi)}$ and $(d, \Sigma) - \text{Mod}(\mathcal{R}_K)^{\text{adm}, (\phi)}$ is quasi-unipotent. In other words, if M is a Taylor admissible Σ -difference equation over \mathcal{R}_K admitting a Frobenius structure, then there exists a finite extension L/K , and an étale extension $\mathcal{R}'_L/\mathcal{R}_L$, such that the scalar extension $M \otimes \mathcal{R}'_L[\log(T)]$ is isomorphic to a direct sum of copies of $\mathcal{R}'_L[\log(T)]$, as Σ -module over $\mathcal{R}'_L[\log(T)]$.*

Proof. Since the Deformation functor is an equivalence it is enough to prove that the deformation of every differential equation M with Frobenius structure is quasi-unipotent. For this we have to prove that $\text{Def}_\Sigma(M)$ is trivialized by some $\mathcal{R}'_K[\log(T)]$, or equivalently that $\text{Def}_\Sigma(M)$ admits a basis of (étale) solutions in $\mathcal{R}'_K[\log(T)]$. We know that the deformation preserves Taylor solutions, the strategy is to prove that the deformation also preserve étale solutions. The proof will need some steps which will be treated in the following sub-sections 6.6.3, 6.6.4, and 6.6.5.

6.6.3 *Reduction to the case of a module of algebraic type.* By [And02, Cor.7.1.6], up to enlarge K , every differential module M with Frobenius structure is direct sum of sub-modules of the form $N \otimes U_m$ where N is trivialized by an étale extension $\mathcal{R}'/\mathcal{R}_K$ (without logarithm) of \mathcal{R}_K , and (U_m, ∇^{U_m}) is the m -dimensional differential module defined by the connection $\nabla^{U_m}(e_i) = T^{-1} \cdot e_{i+1}$, for all $i = 1, \dots, m-1$, and $\nabla^{U_m}(e_m) = 0$. We will say that N is of *algebraic type*, and that U_m is *unipotent*. Since the deformation equivalence preserves this decomposition, we can assume $M = N$ or $M = U_m$. We show first that the deformation of U_m is quasi-unipotent. The analytic cocycle attached to U_m is

$$Y_{U_m}(x, y) = \begin{pmatrix} 1 & \ell_1 & \dots & \ell_{m-2} & \ell_{m-1} \\ 0 & 1 & \ell_1 & \dots & \ell_{m-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & \ell_1 \\ 0 & \dots & 0 & 0 & 1 \end{pmatrix}, \quad (6.18)$$

where $\ell_n := \log(x/y)^n/n!$. Notice that U_m is trivialized by $\mathcal{R}_K[\log(T)]$ since $\tilde{Y} := Y_{U_m}(T, 1) \in GL_m(\mathcal{R}_K[\log(T)])$ is a complete basis of solutions of U_m . By the definition of the deformation, $Y_{U_m}(x, y)$ is also the cocycle attached to $\text{Def}_\Sigma(U_m)$, so the deformation is also trivialized by $\mathcal{R}_K[\log(T)]$. This proves the quasi unipotence of the deformation of U_m .

6.6.4 *A property of the étale solutions.* The unramified extensions of \mathcal{E}_K^\dagger are always obtained as the scalar extension of a special extensions of \mathcal{H}_K^\dagger . We call $\widetilde{\mathcal{H}_K^\dagger} \subset \widetilde{\mathcal{E}_K^\dagger} \subset \widetilde{\mathcal{R}_K}$ the union (in a fixed algebraic closure of $\text{Frac}(\mathcal{R}_K)$) of the special, unramified, and étale extensions of \mathcal{H}_K^\dagger , \mathcal{E}_K^\dagger , and \mathcal{R}_K respectively. Notice that the maximal unramified extension of K^{unr}/K of K is contained in $\widetilde{\mathcal{H}_K^\dagger}$.

LEMMA 6.10. *Let (N, ∇^N) be a differential module with Frobenius structure, with coefficients in \mathcal{R}_K , of algebraic type. Assume that N is defined in the basis \mathbf{e} by the equation $Y' = G(T)Y$. Let $\tilde{Y} := (\tilde{y}_{i,j})_{i,j}$ be its fundamental matrix solution in $\widetilde{\mathcal{R}_{K'}}$ with respect to the basis \mathbf{e} , where K'/K is a convenient finite extension. Then, up to replace K with a finite extension K'/K , the followings statements hold:*

- i) *If $G(T) \in M_n(\mathcal{H}_K^\dagger)$, then \tilde{Y} belongs to $GL_n(\widetilde{\mathcal{H}_K^\dagger})$;*
- ii) *If $G(T) \in M_n(\mathcal{E}_K^\dagger)$, then \tilde{Y} belongs to $GL_n(\widetilde{\mathcal{E}_K^\dagger})$;*
- iii) *Assume that $G(T) \in M_n(\mathcal{E}_K^\dagger)$. Let $(\mathcal{E}_K^\dagger)'$ be the smallest unramified extension of \mathcal{E}_K^\dagger such that $\tilde{Y} \in GL_n((\mathcal{E}_K^\dagger)')$. Then $(\mathcal{E}_K^\dagger)'/\mathcal{E}_K^\dagger$ is generated by the solution \tilde{Y} :*

$$(\mathcal{E}_K^\dagger)' = \mathcal{E}_K^\dagger[\{\tilde{y}_{i,j}\}_{i,j}]; \quad (6.19)$$

By scalar extension the assertion iii) holds also replacing \mathcal{E}_K^\dagger by \mathcal{R}_K .

Proof. [Pul07, Section 8.3]. □

REMARK 6.11. We do not know if point iii) hold also for the special extensions of \mathcal{H}_K^\dagger .

6.6.5 *Quasi unipotence of the deformations of differential modules of algebraic type.* Let now N be a differential equation of algebraic type over \mathcal{R}_K . Let $\text{Can}(N)$ be its canonical extension over \mathcal{H}_K^\dagger , and let $\tilde{Y} \in GL_n(\widetilde{\mathcal{C}_K^{\text{ét}}})$ be the étale solution of $\text{Can}(N)$. The theorem 6.9 will be proved if we show that \tilde{Y} is also a solution of the Σ -difference equation obtained by deformation from $\text{Can}(N)$. If $Y_N(x, y)$ is the analytic cocycle attached to $\text{Can}(N)$, then its Σ -difference equation is defined by the following expression:

$$Y_N(\sigma(x), y) = A_\sigma(x) \cdot Y_N(x, y), \quad \text{for all } \sigma \in \Sigma. \quad (6.20)$$

We have to prove that one has also $\sigma(\tilde{Y}) = A_\sigma(x) \cdot \tilde{Y}$, for all $\sigma \in \Sigma$. The idea is to compare étale and Taylor solutions. For this let $\mathcal{H}_K^\dagger[\tilde{Y}]$ be the \mathcal{H}_K^\dagger -algebra generated by the entries of \tilde{Y} . We have the following

PROPOSITION 6.12. *There exists an \mathcal{H}_K^\dagger -linear injective ring morphism*

$$\text{Tay}_1 : \mathcal{H}_K^\dagger[\tilde{Y}] \longrightarrow \mathcal{A}_{K^{\text{alg}}}(1, 1), \quad (6.21)$$

commuting with the derivation, with Σ , and with the Frobenius, where

$$\mathcal{A}_{K^{\text{alg}}}(1, 1) := \bigcup_{K'/K \text{ finite}} \mathcal{A}_{K'}(1, 1). \quad (6.22)$$

Proof. This inclusion is nothing but the map sending an element of \mathcal{H}_K^\dagger into its Taylor expansion around $T = 1$, and sending the entries of \tilde{Y} into the entries of $Y_N(T, 1)$ (respecting the order of the entries). The existence of this map is provided by the classical differential Galois theory. Indeed, up to replace K by a finite extension K'/K , the differential Galois theory provides a formal isomorphism $\text{Frac}(\mathcal{H}_{K'}^\dagger[\tilde{Y}]) \xrightarrow{\sim} \text{Frac}(\mathcal{H}_{K'}^\dagger[Y_N(T, 1)])$. This map induces the inclusion

$$\text{Tay}_1 : \mathcal{H}_K^\dagger[\tilde{Y}] \subset \mathcal{H}_{K'}^\dagger[\tilde{Y}] \xrightarrow{\sim} \mathcal{H}_{K'}^\dagger[Y_N(T, 1)] \subset \mathcal{A}_{K^{\text{alg}}}(1, 1) \quad (6.23)$$

commuting with the derivation. The fact that Tay_1 commutes also with the Σ and the Frobenius will follow by the uniqueness of Σ and φ in the Special extensions. Since it is not clear whether the special extensions of \mathcal{H}_K^\dagger are generated by the solutions (cf. Remark 6.11), we are obliged to consider \mathcal{E}_K^\dagger and use point iii) of Lemma 6.10. We then proceed as follows. The map (6.23) provides an isomorphism $\text{Tay}_1 : \mathcal{E}_{K'}^\dagger[\tilde{Y}] \xrightarrow{\sim} \mathcal{E}_{K'}^\dagger[Y_N(T, 1)]$. By the point iii) of Lemma 6.10, up to enlarge K' , $\mathcal{E}_{K'}^\dagger[\tilde{Y}]$ is an unramified extension of $\mathcal{E}_{K'}^\dagger$. This last isomorphism commutes then with Σ and φ because of the uniqueness of Σ and φ on unramified extensions of $\mathcal{E}_{K'}^\dagger$.⁹ It follows that also Tay_1 commutes with Σ and φ . \square

Since \tilde{Y} is sent, by the morphism Tay_1 , into a base of solutions of $\text{Can}(N)$ with values in $\mathcal{A}_{K^{\text{alg}}}(1, 1)$, then one has

$$\text{Tay}_1(\tilde{Y}) = Y_N(x, 1) \cdot H, \quad \text{with } H \in GL_n(K^{\text{alg}}). \quad (6.24)$$

This proves that

$$\text{Tay}_1(\sigma(\tilde{Y}) \cdot \tilde{Y}^{-1}) = \sigma(\text{Tay}_1(\tilde{Y})) \cdot \text{Tay}_1(\tilde{Y}^{-1}) = Y_N(\sigma(x), 1) \cdot Y_N(x, 1)^{-1} = A_\sigma(x). \quad (6.25)$$

Since $A_\sigma(x)$ has coefficients in \mathcal{H}_K^\dagger then $\text{Tay}_1^{-1}(A_\sigma(x)) = A_\sigma(x)$, hence $\sigma(\tilde{Y}) = A_\sigma(x) \cdot \tilde{Y}$ as required. This concludes the proof of Theorem 6.9. \square

The previous proof proves in particular the following statement:

COROLLARY 6.13. *Let Σ be a non degenerate family of infinitesimal automorphisms of \mathcal{H}_K^\dagger , as in Theorem 6.9. Let (M, Σ^M) be a (non necessarily admissible) Σ -module over \mathcal{R}_K . Let ∇^M be a connection on M admitting a Frobenius structure. Assume that there exists an étale extension $\mathcal{R}'/\mathcal{R}_K$, such that one of the following three equivalent conditions is verified:*

- i) *There exists a basis of $M \otimes \mathcal{R}'[\log(T)]$ trivializing simultaneously the connection and the action of Σ^M .*

⁹The action of Σ (resp. φ) on $\mathcal{E}_{K'}^\dagger[Y_N(T, 1)]$ induces, via the map Tay_1 , an action on $\mathcal{E}_{K'}^\dagger[\tilde{Y}]$. Since Σ is infinitesimal, the uniformizer π of K' divides $\sigma(T) - T$ (resp. $\varphi(T) - T^p$), and hence this action induces clearly the identity (resp. the p -th power map) on the residual field. This action is hence the unique one of Lemma 6.6.

- ii) *There exists a basis of $M \otimes \mathcal{R}'[\log(T)]$ in which the matrix of the connection is equal to 0 and the matrix of every element of Σ is the identity matrix.*
- iii) *The (d, Σ) -module $M \otimes \mathcal{R}'[\log(T)]$ is isomorphic to a direct sum of the unit object (as (d, Σ) -module over $\mathcal{R}'[\log(T)]$).*

Then (M, Σ^M) is admissible, and coincides with the deformation of (M, ∇^M) :

$$(M, \Sigma^M) = \text{Def}_\Sigma(M, \nabla^M) \quad (6.26)$$

Proof. We consider the Taylor deformation $(M, \Sigma_{\text{Tay}}^M)$ of (M, ∇^M) . By the proof of the above Theorem if $\mathcal{R}'/\mathcal{R}_K$ is an étale extension such that $\mathcal{R}'[\log(T)]$ trivializes (M, ∇^M) , then $\mathcal{R}'[\log(T)]$ trivializes also $(M, \nabla^M, \Sigma_{\text{Tay}}^M)$ as (d, Σ) -module. In other words the basis trivializing the connection trivializes also the action of Σ_{Tay}^M . We have then two actions of Σ on M , namely (M, Σ^M) , and $(M, \Sigma_{\text{Tay}}^M)$, which have the property that, after scalar extension to $\mathcal{R}'[\log(T)]$, there exists a base change trivializing simultaneously these actions. The actions coincides after base change, hence they are equal in the original base. So we have $(M, \Sigma^M) = (M, \Sigma_{\text{Tay}}^M)$. \square

SECOND PART: APPLICATIONS

7. Differential equations and (φ, Γ) -modules over the Robba ring.

We firstly fix very quickly some notations about the theory of (φ, Γ_K) -modules. The infinitesimality and non degeneracy of the action of (a subgroup of) Γ_K will follows from a lemma of [Col] (cf. Lemma 7.5), for this reason we will follows its notations. For further details we refer to [Col].

Let K/\mathbb{Q}_p be a finite extension with residual field k , and let $F := \mathbf{W}(k)[1/p]$ its absolutely unramified subfield. Let K_n/K be the field generated over K by the p^n th root of unity, and let $K_\infty := \cup_n K_n$. Let $\mathcal{G}_K := \text{Gal}(K^{\text{alg}}/K)$, $\mathcal{H}_K := \text{Gal}(K^{\text{alg}}/K_\infty)$, and $\Gamma_K = \mathcal{G}_K/\mathcal{H}_K = \text{Gal}(K_\infty/K)$. The cyclotomic character $\chi : \mathcal{G}_K \rightarrow \mathbb{Z}_p^*$ factorizes through Γ_K , and identifies it with a subgroup of \mathbb{Z}_p^* with finite index. For $\gamma \in \Gamma_K$ we set $n(\gamma) := v_p(\chi(\gamma) - 1)$, where v_p denotes the p -adic valuation normalized by $v_p(p) = 1$. The filtration of Γ_K induced by χ will be denoted by

$$\Gamma_K^{(n)} := \{ \gamma \in \Gamma_K \mid n(\gamma) \geq n \} = \chi^{-1}(1 + p^n \mathbb{Z}_p). \quad (7.1)$$

We denote by $c(K)$ the conductor of K . If \mathcal{G}_F^s denotes the upper numbering ramification subgroup of \mathcal{G}_F (where as usual $\mathcal{G}_F = \mathcal{G}_F^{-1}$, \mathcal{G}_F^s is equal to the inertia subgroup for all $s \in]-1, 0]$, and the wild inertia is given by $\cup_{s>0} \mathcal{G}_F^s$), then $c(K) = \sup(s \mid K^{\mathcal{G}_F^s} = L)$ where $K^{\mathcal{G}_F^s}$ denotes the subfield of K formed by the fixed points by the action of \mathcal{G}_F^s . One has (cf. [Col, Lemme 4.2])

$$\Gamma_K^{(n)} = \Gamma_{K_n}, \quad \text{for all } n \geq [c(K) + 1], \quad (7.2)$$

where $[x]$ denotes the smaller integer greater than or equal to the real number x ($[c(K) + 1]$ is denoted by $n_0(K)$ in [Col, Section 4.1]). For all extension A/B of complete discrete valued fields we denote by $\mathfrak{d}_{A/B}$ its different.

7.0.6 Some notations about the field of norms. For all finite extension K/F we denote by \mathbf{E}_K the field of norms of K (cf. [Col, Section 4.3]). \mathbf{E}_K is a finite extension of \mathbf{E}_F of degree $[K_\infty : F_\infty]$. We denote by $v_{\mathbf{E}} : \mathbf{E}_K \rightarrow \mathbb{R}$ the discrete valuation defined by $v_{\mathbf{E}}((x^{(n)})_n) = v_p(x^{(0)})$, where as usual $(x^{(n)})_{n \geq 0} \in \mathbf{E}_K$ is a sequence verifying $(x^{(n+1)})^p = x^{(n)}$, for all $n \geq 0$. Let $\bar{\pi} := \varepsilon - 1$, where $\varepsilon = (1, \varepsilon^{(1)}, \varepsilon^{(2)}, \dots)$ verifies $\varepsilon^{(1)} \neq 1$, then $v_{\mathbf{E}}(\bar{\pi}) = \frac{p}{p-1}$. We denote by $\bar{\pi}_K$ a uniformizer element of \mathbf{E}_K , and by $\delta_K := v_{\mathbf{E}}(\mathfrak{d}_{\mathbf{E}_K/\mathbf{E}_F})$. By [Col, Prop.4.12] one has $v_{\mathbf{E}}(\mathfrak{d}_{\mathbf{E}_K/\mathbf{E}_F}) = \lim_{n \rightarrow +\infty} p^n v_p(\mathfrak{d}_{K_n/F_n}) \leq$

$\frac{1}{p-1}p^{[c(K)+1]}$. Finally we set

$$r_K := \begin{cases} (2v_{\mathbf{E}}(\mathfrak{d}_{\mathbf{E}_K/\mathbf{E}_F}))^{-1} & \text{if } \mathbf{E}_K/\mathbf{E}_F \text{ is ramified,} \\ 1 & \text{if } \mathbf{E}_K/\mathbf{E}_F \text{ is unramified.} \end{cases} \quad (7.3)$$

7.0.7 Some notations about overconvergent functions. Following the notations of [Col] we consider the functor associating to every finite extension K/F the rings $\mathbf{B}_K^{(0,r]}$, $\mathbf{B}_K^{[0,r]}$, $r > 0$, and $\mathbf{B}_K^\dagger := \cup_{r>0} \mathbf{B}_K^{(0,r]}$, $\mathbf{B}_{K,\text{rig}}^\dagger := \cup_{r>0} \mathbf{B}_K^{[0,r]}$, (whose definition can be founded in [Col, Section 7.3]). These rings admit the following description in terms of analytic functions. Let $\rho_r := |p|^{r \cdot v_{\mathbf{E}}(\bar{\pi}_K)}$. For $r < r_K$ the ring $\mathbf{B}_K^{[0,r]}$ (resp. $\mathbf{B}_K^{(0,r]}$) is isomorphic to the ring of (resp. bounded) analytic functions converging in the annulus $\rho_r \leq |T| < 1$, with coefficients in the field $F' := \mathbf{W}(k_\infty)[1/p]$, where k_∞ is the residual field of K_∞ (cf. [Col, Prop. 7.5, and Prop. 7.6]):

$$\mathbf{B}_K^{(0,r]} \xrightarrow{\sim} \mathcal{B}_{F'}([\rho_r, 1[), \quad (7.4)$$

$$\mathbf{B}_K^{[0,r]} \xrightarrow{\sim} \mathcal{A}_{F'}([\rho_r, 1[). \quad (7.5)$$

Here the word bounded means that $\mathcal{B}_{F'}([\rho_r, 1[)$ is defined as the sub-ring of $\mathcal{A}_{F'}([\rho_r, 1[)$ of the functions $f(T)$ satisfying $\sup_{\rho \in [\rho_r, 1[} |f|_\rho < \infty$.

Maintaining the assumption $r < r_K$, by the above isomorphisms (7.4) and (7.5), the valuation $v^{(0,r]}$ on $\mathbf{B}_K^{(0,r]}$ and $\mathbf{B}_K^{[0,r]}$ (defined in [Col]) corresponds to the norm $|\cdot|_{\rho_r}^{1/r}$ of $\mathcal{A}_{F'}([\rho_r, 1[)$ (cf. [Col, Prop.7.5]). Analogously one has

$$\mathbf{B}_K^\dagger := \cup_{r>0} \mathbf{B}_K^{(0,r]} \xrightarrow{\sim} \mathcal{E}_{F'}^\dagger, \quad (7.6)$$

$$\mathbf{B}_{K,\text{rig}}^\dagger := \cup_{r>0} \mathbf{B}_K^{[0,r]} \xrightarrow{\sim} \mathcal{R}_{F'}. \quad (7.7)$$

7.0.8 Actions of Γ_K and φ . All the above rings have a canonical action of Γ_K . The Frobenius φ acts on \mathbf{B}_K^\dagger and $\mathbf{B}_{K,\text{rig}}^\dagger$. If L/K is a finite extension then the action of Γ_L on $\mathbf{B}_{\text{rig},L}^\dagger$ stabilizes $\mathbf{B}_{\text{rig},K}^\dagger \subseteq \mathbf{B}_{\text{rig},L}^\dagger$, and coincides with its action through Γ_K by the canonical inclusion $\Gamma_L \subseteq \Gamma_K$ (whose image is equal to $\Gamma_K^{(n)}$, for some $n \geq 0$).

Let ℓ_T be a variable, one extends the actions of Γ_K and φ to $\mathbf{B}_{\text{rig},K}^\dagger[\ell_T]$ by setting $\varphi(\ell_T) := p \cdot \ell_T$, and $\gamma(\ell_T) := \ell_T + \log(\gamma(T)/T)$ (cf. Lemma 6.5), where T is the variable on $\mathbf{B}_{\text{rig},K}^\dagger$.

7.1 From De Rham representations to differential equations over the Robba ring

In this section we recall briefly the todays standard relations between p -adic representations, (φ, Γ_K) -modules, and differential equations over the Robba ring $\mathcal{R}_{F'}$. In [Fon90] J.M. Fontaine obtained an equivalence of categories, indicated by D, between the category $\text{Rep}(\mathcal{G}_K)$ of all continuous p -adic representations of \mathcal{G}_K (i.e. finite dimensional \mathbb{Q}_p -vector spaces together a continuous action of \mathcal{G}_K), and the category of the so called étale (φ, Γ_K) -modules over the fraction field $\mathcal{E}_{F'}$ of a Cohen ring $\mathcal{O}_{\mathcal{E}_{F'}}$ of \mathbf{E}_K :

$$\text{D} : \text{Rep}(\mathcal{G}_K) \xrightarrow{\sim} (\varphi, \Gamma_K) - \text{Mod}(\mathcal{E}_{F'})^{\text{ét}}. \quad (7.8)$$

The objects of the target category are finite free modules over $\mathcal{E}_{F'}$ together with a semilinear and continuous actions of Γ_K and of φ verifying i) the actions of φ and Γ_K commute between them, ii) there exists a lattice over $\mathcal{O}_{\mathcal{E}_{F'}}$ stable under φ and Γ_K , iii) the matrix of φ in a basis of the lattice is invertible, i.e. the image of φ generates the lattice.

The elements of $\mathcal{E}_{F'}$ can be described as series $\sum_{i \in \mathbb{Z}} a_i T^i$, $a_i \in F'$, satisfying $\sup |a_i| < \infty$, and $\lim_{i \rightarrow -\infty} |a_i| = 0$. This ring is somewhat too big because its elements do not converge anywhere. A

fundamental result of F.Cherbouner and P.Colmez (cf. [CC98]) permits to find the same equivalence replacing $\mathcal{E}_{F'}$ by $\mathbf{B}_K^\dagger \xrightarrow{\sim} \mathcal{E}_{F'}^\dagger$:

THEOREM 7.1 [CC98, III 5.2]. *Let V be a p -adic representation of \mathcal{G}_K . The family of sub- $\mathcal{E}_{F'}^\dagger$ -modules of $D(V)$ which are of finite type, and stable under φ and Γ_K , admits a biggest element $D^\dagger(V)$ verifying $D(V) = D^\dagger(V) \otimes_{\mathcal{E}_{F'}^\dagger} \mathcal{E}_{F'}$. The functor $V \mapsto D^\dagger(V)$ is an equivalence and one has the commutative diagram*

$$\begin{array}{ccc} \text{Rep}(\mathcal{G}_K) & \xrightarrow[\sim]{D} & (\varphi, \Gamma_K) - \text{Mod}(\mathcal{E}_{F'})^{\text{ét}} \\ & \searrow_{D^\dagger} & \uparrow \wr \\ & & (\varphi, \Gamma_K) - \text{Mod}(\mathcal{E}_{F'}^\dagger)^{\text{ét}} \end{array} \quad (7.9)$$

where the vertical equivalence is the scalar extension functor from $\mathcal{E}_{F'}^\dagger$ to $\mathcal{E}_{F'}$. \square

In this context, L.Berger was able to deduce, from the action of Γ_K on $D^\dagger(V)$, the existence of a connection ∇_V on $D_{\text{rig}}^\dagger(V) = D^\dagger(V) \otimes_{\mathcal{E}_{F'}^\dagger} \mathcal{R}_{F'}$. We denote by t an element of $\mathcal{R}_{F'}$ on which Γ_K act as $\gamma(t) = \chi(\gamma)t$, and $\varphi(t) = pt$. The connection ∇_V is defined by the following limit (cf. [Ber02, Section 5.1])

$$\nabla_V := \lim_{\gamma \rightarrow 1} \frac{\gamma - 1}{\chi(\gamma) - 1} \quad (7.10)$$

and verifies the Leibnitz rule with respect to the derivation $\nabla := \lim_{\gamma \rightarrow 1} \frac{\gamma - 1}{\chi(\gamma) - 1}$ of $\mathcal{R}_{F'}$.

REMARK 7.2. If $K = F$, then $\gamma(T) = (1 + T)^{\chi(\gamma)} - 1$, $\varphi(T) = (1 + T)^p - 1$, and $t = \log(1 + T)$. Then $\nabla = (1 + T) \log(1 + T) \frac{d}{dT}$.

This connection presents singularities. Under the assumption that the starting representation V is of de Rham type, L.Berger is able to desingularize the connection ∇_V obtaining a differential equation over the Robba ring. More precisely one proves the existence of a unique $\mathcal{R}_{F'}$ -lattice of $D_{\text{rig}}^\dagger(V) \otimes_{\mathcal{R}_{F'}} \mathcal{R}_{F'}[t^{-1}]$ stable under the connection $t^{-1}\nabla_V$, under the Frobenius φ , and under the action of Γ_K . One obtains hence a differential equation $N_{dR}(V)$ over $\mathcal{R}_{F'}$. This construction is functorial and one has a faithful functor (cf. [Ber02, Th. 5.20])

$$N_{dR} : \text{Rep}_{dR}(\mathcal{G}_K) \longrightarrow d - \text{Mod}(\mathcal{R}_{F'})^{(\varphi)}, \quad (7.11)$$

where $\text{Rep}_{dR}(\mathcal{G}_K)$ is the category of de Rham representations of \mathcal{G}_K , and $d - \text{Mod}(\mathcal{R}_{F'})^{(\varphi)}$ is the category of differential equations over $\mathcal{R}_{F'}$ admitting an unspecified action of φ .

Recently L.Berger have been able to make this result more precise (cf. [Ber07]). The differential equation $N_{dR}(V)$ arises naturally with an action of φ and Γ_K . Let $(\varphi, \Gamma_K) - \text{Mod}(\mathcal{R}_{F'})$ the category of finite free $\mathcal{R}_{F'}$ -modules D together with an action of φ , such that $\varphi(D)$ generates D , and an action of Γ_K commuting with φ . Denote by $(\varphi, \Gamma_K) - \text{Mod}(\mathcal{R}_{F'})^{\text{ét}, \text{LT}}$ the fully faithful subcategory of $(\varphi, \Gamma_K) - \text{Mod}(\mathcal{R}_{F'})$ formed by objects such that the action of φ is *étale* in the sense of [Ber07, Def. IV.1.2], and such that the Lie algebra of Γ_K acts *locally trivially* in the sense of [Ber07, Def. III.1.2]. These assumptions provide that the connection ∇_V can be constructed from the action of Γ_K , and desingularized as in [Ber02]. Then the above functor (7.11) can actually be refined into an equivalence that we still call D^\dagger

$$D^\dagger : \text{Rep}_{dR}(\mathcal{G}_K) \xrightarrow{\sim} (\varphi, \Gamma_K) - \text{Mod}(\mathcal{R}_{F'})^{\text{ét}, \text{LT}}. \quad (7.12)$$

If now $(\varphi, \nabla) - \text{Mod}(\mathcal{R}_{F'})^{\text{ét}}$ denotes the category of (finite free) differential $\mathcal{R}_{F'}$ -modules together with an étale action of φ (in the sense of [Ber07, Def.IV.1.2]) compatible with the derivation in

which the morphisms are assumed to commute with the connection and with φ , then Berger is able to construct a functor

$$C_{\Gamma_K} : (\varphi, \Gamma_K) - \text{Mod}(\mathcal{R}_{F'})^{\text{ét,LT}} \longrightarrow (\varphi, \nabla) - \text{Mod}(\mathcal{R}_{F'})^{\text{ét}}, \quad (7.13)$$

where the connection is constructed from the action of Γ_K as in [Ber02]. The functor N_{dR} is then the composite of these functors with the evident functor $(\varphi, \nabla) - \text{Mod}(\mathcal{R}_{F'})^{\text{ét}} \longrightarrow d - \text{Mod}(\mathcal{R}_{F'})^{(\varphi)}$.

7.2 Γ_K -Deformation of solvable differential equations

In this subsection we apply theorem 6.1 to deform solvable differential equations over $\mathcal{R}_{F'}$ into admissible $\Gamma_K^{(n)}$ -modules, for some $n \geq n_K$ (see below the definition of n_K).

REMARK 7.3. Notice that we can not apply the results of the previous section about the quasi-unipotence. Indeed, in the case in which the action of Γ_K is defined over $\mathcal{H}_{F'}^\dagger$,¹⁰ the action of Γ_K is not infinitesimal on $\mathcal{H}_{F'}^\dagger$. Indeed the reduction of this action on the residual ring $k_\infty[t, t^{-1}]$ of $\mathcal{H}_{F'}^\dagger$ (and hence on the residual ring $k_\infty((t))$ of $\mathcal{E}_{F'}^\dagger$) is not equal to the identity. The above methods fails because we do not have a way to compare Taylor solutions with étale solutions (cf. Proposition 6.12), because none of the subgroups $\Gamma_K^{(n)}$ of Γ_K acts on the algebra $\mathcal{A}_{K^{\text{alg}}}(1, 1)$. Nevertheless the action of a subgroup of Γ_K will be infinitesimal over $\mathcal{R}_{F'}$.

7.2.1 *Infinitesimality and non degeneracy of the action of $\Gamma_K^{(n)}$, for $n \geq n_K$.* Here again we denote by $\lceil x \rceil$ denotes the smaller integer greater than or equal to the real number x . Let n_K be the smallest integer satisfying $n_K \geq \lceil c(K) + 1 \rceil$ and $n_K > \log_p\left(\frac{v_{\mathbf{E}}(\bar{\pi}_K) + \delta_K}{v_{\mathbf{E}}(\bar{\pi})}\right)$ (if $K = F$, then $n_K = 1$).

LEMMA 7.4. *If $n \geq n_K$, the action of $\Gamma_K^{(n)}$ is infinitesimal and non degenerate on $\mathcal{R}_{F'}$ and $\mathcal{A}_{F'}(\lceil \rho, 1 \rceil)$, for all $0 < \rho < 1$ sufficiently close to 1.*

Proof. The infinitesimality follows immediately from [Col, Lemme 9.4] which we reproduce here for the convenience of the reader:

LEMMA 7.5 ([Col, Lemma 9.4]). *Let $\gamma \in \Gamma_K$ be such that $n(\gamma) \geq \lceil c(K) + 1 \rceil$. Then for all $\rho < 1$ satisfying $|p|^{v_{\mathbf{E}}(\bar{\pi}_K) \cdot \inf(r_K, p^{-n(\gamma)})} < \rho < 1$ one has*

$$|\gamma(T) - T|_\rho = \rho^{s(\gamma)}. \quad (7.14)$$

where

$$s(\gamma) := v_{\mathbf{E}}(\bar{\pi}_K)^{-1} \cdot (p^{n(\gamma)} \cdot v_{\mathbf{E}}(\bar{\pi}) - \delta_K). \quad (7.15)$$

Proof. The above statement is a reformulation of [Col, Prop.7.5 and Prop.7.6]. The notations have been explained in section 7.0.7 ($\lceil c(K) + 1 \rceil$ is denoted by $n_0(K)$ in [Col, Section 4.1]). \square

We recall that the automorphism γ is infinitesimal if and only if $|\gamma(T) - T|_\rho < \rho$ for $\rho \in]1 - \varepsilon, 1[$ (cf. Remark 5.25). This condition becomes $v_{\mathbf{E}}(\bar{\pi}_K)^{-1} \cdot (p^{n(\gamma)} \cdot v_{\mathbf{E}}(\bar{\pi}) - \delta_K) > 1$, and hence $n(\gamma) > \log_p\left(\frac{v_{\mathbf{E}}(\bar{\pi}_K) + \delta_K}{v_{\mathbf{E}}(\bar{\pi})}\right)$. The non degeneracy is then a direct consequence of (7.14) by Lemma 5.13. This concludes the proof of Lemma 7.4. \square

REMARK 7.6. If $K = F$ then the action of $\Gamma_K^{(n)}$ is infinitesimal and non degenerate for all $n \geq 1$.

¹⁰For example if $K = F$ we have $\gamma(T) = (T + 1)^{x(\gamma)} - 1$

7.2.2 *Deformation of the action of Γ_K .* By Lemma 7.4, we can apply Theorem 6.1 to each family $\Sigma = \Gamma_K^{(n)}$, for all $n \geq n_K$. We summarize it in the following

COROLLARY 7.7 (Deformation of the action of Γ_K). *For all $n \geq n_K$, the category $d\text{-Mod}(\mathcal{R}_K)^{\text{adm}(\Gamma_K^{(n)})}$ is equivalent to $\Gamma_K^{(n)}\text{-Mod}(\mathcal{R}_K)^{\text{adm}}$.* \square

We now want to describe the class of differential equations belonging to $d\text{-Mod}(\mathcal{R}_K)^{\text{adm}(\Gamma_K^{(n)})}$.

LEMMA 7.8. *Let $\gamma \in \Gamma_K^{(n)}$, $\gamma \neq 1$, $n \geq n_K$. Let $Y(x, y)$ be a γ -admissible analytic cocycle over \mathcal{R}_K , then $Y(x, y)$ is solvable.*

Proof. By definition of the radius one has $\text{Rad}(Y(x, y), \rho) \leq \rho$, for all ρ sufficiently close to 1. The admissibility condition is $\text{Rad}(Y(x, y), \rho) > |\delta_\gamma(T)|_\rho = |\gamma(T) - T|_\rho$, for all ρ close to 1. By Lemma 7.4 $|\delta_\gamma(T)|_\rho$ is known for ρ sufficiently close to 1^- , so $1 \geq \lim_{\rho \rightarrow 1^-} \text{Rad}(Y(x, y), \rho) \geq \lim_{\rho \rightarrow 1^-} |\gamma(T) - T|_\rho = 1$. \square

PROPOSITION 7.9. *For all $n \geq n_K$, one has*

$$d\text{-Mod}(\mathcal{R}_K)^{\text{adm}(\Gamma_K^{(n)})} = d\text{-Mod}(\mathcal{R}_K)^{\text{Sl} < s(K, n)}, \quad (7.16)$$

where

$$s(K, n) := \frac{p^{n+1} \cdot v_{\mathbf{E}}(\bar{\pi}) + \delta_K}{v_{\mathbf{E}}(\bar{\pi}_K)} - 1. \quad (7.17)$$

In other words a solvable differential equation is $\Gamma_K^{(n)}$ -admissible if and only if its p -adic slope is strictly less than $s(K, n)$.

Proof. We recall that a solvable cocycle $Y(x, y)$ over \mathcal{R}_K has a radius of convergence of the form $\text{Rad}(Y(x, y), \rho) = \rho^\beta$, for all $\rho \in I_\varepsilon$, for $\varepsilon > 0$ sufficiently small, where the number $\beta - 1$ is the p -adic slope of the cocycle (cf. section 6.2). The $\Gamma_K^{(n)}$ compatibility of $Y(x, y)$ is $\text{Rad}(Y(x, y), \rho) > |\gamma(T) - T|_\rho$ for all ρ sufficiently close to 1^- . The condition is hence $\rho^\beta > \rho^{s(\gamma)}$ for all $\gamma \in \Gamma_K^{(n)}$. In other words $\beta - 1 < \min_{n(\gamma) \geq n} (s(\gamma) - 1) = \frac{p^{n+1} \cdot v_{\mathbf{E}}(\bar{\pi}) + \delta_K}{v_{\mathbf{E}}(\bar{\pi}_K)} - 1$. \square

REMARK 7.10. One has $s(F, n) = p^n - 1$.

7.2.3 *Germ of admissible Γ_K -actions, and solvable differential equations over $\mathcal{R}_{F'}$.* In order to take in account the largest class of differential equations, we introduce the following notion

DEFINITION 7.11. A *Germ of admissible Γ_K -action* is a finite free module over $\mathcal{R}_{F'}$, together with an *Taylor admissible* semilinear action of a subgroup $\Gamma_K^{(n)}$, for some unspecified $n \geq 1$. A morphism between germs of admissible Γ_K -actions is an $\mathcal{R}_{F'}$ -linear map commuting with the action of $\Gamma_K^{(n)}$ for n sufficiently large. We denote by $\text{Germ}(\Gamma_K)^{\text{adm}}$ the category of germs of admissible Γ_K -actions.

One verifies that

$$\text{Germ}(\Gamma_K)^{\text{adm}} = \bigcup_{n \geq n_K} \Gamma_K^{(n)}\text{-Mod}(\mathcal{R}_{F'})^{\text{adm}}. \quad (7.18)$$

Since $\lim_{n \rightarrow +\infty} s(K, n) = +\infty$, by Proposition 7.9, we have the following

THEOREM 7.12. *The category $d\text{-Mod}(\mathcal{R}_{F'})^{[1]}$ of solvable differential equations is equivalent to the category $\text{Germ}(\Gamma_K)^{\text{adm}}$.*

Proof. The theorem follows by passing to the limit the Proposition 7.9, since $\lim_{n \rightarrow +\infty} s(K, n) = +\infty$. Indeed clearly one has $d\text{-Mod}(\mathcal{R}_{F'})^{[1]} = \bigcup_{n \geq 1} d\text{-Mod}(\mathcal{R}_{F'})^{\text{Sl} < s(K, n)}$. On the other hand, by Proposition 7.9 one has $d\text{-Mod}(\mathcal{R}_{F'})^{\text{Sl} < s(K, n)} = d\text{-Mod}(\mathcal{R}_{F'})^{\text{adm}(\Gamma_K^{(n)})}$. Now by Deformation one has $d\text{-Mod}(\mathcal{R}_{F'})^{\text{adm}(\Gamma_K^{(n)})} \xrightarrow{\sim} \Gamma_K^{(n)}\text{-Mod}(\mathcal{R}_{F'})^{\text{adm}}$. One concludes then by the equality (7.18). \square

7.3 Taylor admissibility of (φ, Γ_K) -modules arising from de Rham representations

For all $\beta \geq 0$ we set

$$n(\beta) := \min\{n \mid n \geq n_K \text{ and } \beta < s(K, n)\}. \quad (7.19)$$

For n sufficiently large the differential equation $N_{dR}(V)$ has two actions of $\Gamma_K^{(n)}$. The first one arises from the theory of L.Berger. The second one is deduced by deformation from the differential equation itself using theorem 6.1. More precisely if the p -adic slope of $N_{dR}(V)$ is equal to β , then by Corollary 7.7, and Proposition 7.9, by deformation we recover an action of $\Gamma_K^{n(\beta)}$ on $N_{dR}(V)$. In this subsection we prove the following

THEOREM 7.13. *There exists $n \geq n_K$ such that the action of $\Gamma_K^{(n)}$ on $N_{dR}(V)$ arising by the theory of L.Berger coincides with the action of $\Gamma_K^{(n)}$ deduced by deformation from the differential equation $N_{dR}(V)$ by Corollary 7.7. In particular the connection $N_{dR}(V)$ together with the action of Γ_K (arising from the theory of Berger) lies in $\text{Germ}(\Gamma_K)^{\text{adm}}$.*

Proof. By [Mar04, Prop.5.2] there exists a finite extension L/K such that $N_{dR,L}(V) := N_{dR}(V) \otimes_{\mathbf{B}_{\text{rig},K}^\dagger} \mathbf{B}_{\text{rig},L}^\dagger[\ell_T]$ admits a basis in which the connection and the action of Γ_L on $N_{dR,L}(V)$ arising from the theory of Berger are both trivial (here T denotes the variable of $\mathcal{R}_{F'} = \mathbf{B}_{\text{rig},K}^\dagger$). We consider moreover $r > 0$ sufficiently small such that the differential equation $N_{dR}(V)$ is Γ_K -admissible over the annulus $\mathbf{B}_K^{[0,r]}$. We consider then an $r' > 0$ such that the connection $N_{dR,L}(V)$ and the action of Γ_L are defined on $\mathbf{B}_L^{[0,r']} \subset \mathbf{B}_{\text{rig},L}^\dagger$ and are both trivial over $\mathbf{B}_L^{[0,r']}$. By (7.7) the ring $\mathbf{B}_L^{[0,r']}$ (resp. $\mathbf{B}_K^{[0,r]}$) is isomorphic to a ring of analytic functions over an annulus $\mathcal{C}(\rho_L, 1[)$ (resp. $\mathcal{C}(\rho_K, 1[)$) with coefficients in a field F'_L (resp. $F'_K = F'$). We can chose $r' > 0$ sufficiently small in order to have the natural inclusion $\mathbf{B}_K^{[0,r]} \subseteq \mathbf{B}_L^{[0,r']}$ which corresponds to an analytic map $\pi_K^L : \mathcal{C}(\rho_L, 1[) \rightarrow \mathcal{C}(\rho_K, 1[)$.

Now for all $\rho_K < \rho < 1$ consider the disk $D^-(t_\rho, R_\rho) \subset \mathcal{C}(\rho_K, 1[)$, where t_ρ is the Dwork generic point corresponding to $|\cdot|_\rho$, and $R_\rho := \text{Rad}(N_{dR}(V), \rho)$. Let $D^-(t_L, R_{L,\rho})$ be a disk contained in the inverse image of $D^-(t_\rho, R_\rho)$ by π_K^L . Now assume n sufficiently large in order that $\Gamma_L^{(n)}$ stabilizes $D^-(t_L, R_{L,\rho})$. Notice that $\Gamma_L^{(n)}$ is canonically identified with $\Gamma_K^{(m)}$ for some $m \geq n$. We have then the following diagrams

$$\begin{array}{ccc} D^-(t_L, R_{L,\rho}) & \subset & \mathcal{C}(\rho_L, 1[) \\ \pi_K^L \downarrow & & \downarrow \pi_K^L \\ D^-(t_\rho, R_\rho) & \subset & \mathcal{C}(\rho_K, 1[) \end{array} \quad ; \quad \begin{array}{ccccc} \mathcal{A}_{F'_L}(t_L, R_{L,\rho}) & \longleftarrow & \mathbf{B}_L^{[0,r']} & \longleftarrow & \mathbf{B}_{\text{rig},L}^\dagger \\ \downarrow (\pi_K^L)^* & & \downarrow (\pi_K^L)^* & & \downarrow (\pi_K^L)^* \\ \mathcal{A}_{F'}(t_\rho, R_\rho) & \longleftarrow & \mathbf{B}_K^{[0,r]} & \longleftarrow & \mathbf{B}_{\text{rig},K}^\dagger \end{array} \quad (7.20)$$

Now the inclusion $D^-(t_L, R_{L,\rho}) \subset \mathcal{C}(\rho_L, 1[)$ identifies the Taylor solutions in $GL_n(\mathcal{A}_{F'_L}(t_L, R_{L,\rho}))$ of the differential equation $N_{dR}(V)$ with a basis of étale solutions $\tilde{Y} \in GL_n(\mathbf{B}_{\text{rig},L}^\dagger[\ell_T])$. Moreover the inclusion $\mathcal{A}_{F'}(t_\rho, R_\rho) \rightarrow \mathcal{A}_{F'_L}(t_L, R_{L,\rho})$ induced by π_K^L identifies the Taylor solution of $N_{dR}(V)$ in $GL_n(\mathcal{A}_{F'}(t_\rho, R_\rho))$ with its Taylor solution over $\mathcal{A}_{F'_L}(t_L, R_{L,\rho})$. The group $\Gamma_L^{(n)}$ acts on all these rings and its action on $\mathcal{A}_{F'}(t_\rho, R_\rho)$, $\mathbf{B}_K^{[0,r]}$, $\mathbf{B}_{\text{rig},K}^\dagger$ coincides with that induced by its inclusion in Γ_K (cf. section 7.0.8). This proves that the action of $\Gamma_L^{(n)}$ on the étale solutions \tilde{Y} coincides with its action on the Taylor solutions in $GL_n(\mathcal{A}_{F'}(t_\rho, R_\rho))$ which is the action obtained by deformation from the connection $N_{dR}(V)$. Hence the action of $\Gamma_L^{(n)}$ on $N_{dR}(V)$ arising by the theory of L.Berger coincides with the action of $\Gamma_K^{(n)}$ deduced by deformation from the differential equation $N_{dR}(V)$. \square

7.4 From differential equation to de Rham representations

We denote by $\text{Germ Rep}_{dR}(\mathcal{G}_K)$ the category $\varinjlim_{n \geq 0} \text{Rep}_{dR}(\mathcal{G}_{K_n})$. For all $n \geq 0$ the equivalence $\text{Rep}_{dR}(\mathcal{G}_{K_n}) \xrightarrow{\sim} (\varphi, \Gamma_{K_n}) - \text{Mod}(\mathcal{R}_{F'})^{\text{ét}, \text{LT}}$ (cf. (7.12)) commutes with the restrictions $\text{Rep}_{dR}(\mathcal{G}_{K_n}) \rightarrow \text{Rep}_{dR}(\mathcal{G}_{K_{n+1}})$ and $(\varphi, \Gamma_{K_n}) - \text{Mod}(\mathcal{R}_{F'})^{\text{ét}, \text{LT}} \rightarrow (\varphi, \Gamma_{K_{n+1}}) - \text{Mod}(\mathcal{R}_{F'})^{\text{ét}, \text{LT}}$. So we have an equivalence $\text{Germ Rep}_{dR}(\mathcal{G}_K) \xrightarrow{\sim} \text{Germ}(\varphi, \Gamma_K) - \text{Mod}(\mathcal{R}_{F'})^{\text{ét}, \text{LT}}$, where $\text{Germ}(\varphi, \Gamma_K) - \text{Mod}(\mathcal{R}_{F'})^{\text{ét}, \text{LT}} := \varinjlim_{n \geq 0} (\varphi, \Gamma_{K_n}) - \text{Mod}(\mathcal{R}_{F'})^{\text{ét}, \text{LT}}$. On the other hand the functor C_{Γ_K} (cf. (7.13)) extends clearly to $\text{Germ}(\varphi, \Gamma_K) - \text{Mod}(\mathcal{R}_{F'})^{\text{ét}, \text{LT}}$ so we have the following diagram

$$\begin{array}{ccc}
 \text{Germ}(\varphi, \Gamma_K) - \text{Mod}(\mathcal{R}_{F'})^{\text{ét}} & \supseteq & \text{Germ}(\varphi, \Gamma_K) - \text{Mod}(\mathcal{R}_{F'})^{\text{ét}, \text{adm}} \\
 \cup & \nearrow \subseteq & \uparrow \text{Def}_{\Gamma_K} \\
 \text{Germ Rep}(\mathcal{G}_K) \xrightarrow[\sim]{D^\dagger} \text{Germ}(\varphi, \Gamma_K) - \text{Mod}(\mathcal{R}_{F'})^{\text{ét}, \text{LT}} & \xrightarrow{C_{\Gamma_K}} & (\varphi, \nabla) - \text{Mod}(\mathcal{R}_{F'})^{\text{ét}} \\
 & \searrow N_{dR} & \downarrow \\
 & & d - \text{Mod}(\mathcal{R}_{F'})^{(\varphi)}
 \end{array} \tag{7.21}$$

where $\text{Germ}(\varphi, \Gamma_K) - \text{Mod}(\mathcal{R}_{F'})^{\text{ét}, \text{adm}}$ denotes the fully faithful sub-category of $\text{Germ}(\varphi, \Gamma_K) - \text{Mod}(\mathcal{R}_{F'})^{\text{ét}}$ whose objects are admissible, and where the inclusion

$$\text{Germ}(\varphi, \Gamma_K) - \text{Mod}(\mathcal{R}_{F'})^{\text{ét}, \text{LT}} \subseteq \text{Germ}(\varphi, \Gamma_K) - \text{Mod}(\mathcal{R}_{F'})^{\text{ét}, \text{adm}} \tag{7.22}$$

is provided by Theorem 7.13.

THEOREM 7.14. *The above diagram commutes. In particular the functors C_{Γ_K} and N_{dR} can be described as*

$$C_{\Gamma_K} = \text{Conf}_{\Gamma_K}, \quad \text{and} \quad N_{dR} = \text{Conf}_{\Gamma_K} \circ D^\dagger, \tag{7.23}$$

where $\text{Conf}_{\Gamma_K} := (\text{Def}_{\Gamma_K})^{-1}$.

Proof. Let (M, φ, Γ_K) be an object of $\text{Germ}(\varphi, \Gamma_K) - \text{Mod}(\mathcal{R}_{F'})^{\text{ét}, \text{LT}}$. By [Mar04, Prop.5.2], there exists an étale extension $B_{L, \text{rig}}^\dagger[\ell_T]$ trivializing simultaneously the action of a subgroup of Γ_K , and the connection of $C_{\Gamma_K}(M)$. By the proof of Theorem 7.13 also the action of (a subgroup of) Γ_K obtained by deformation from the connection $C_{\Gamma_K}(M)$ is trivialized by $B_{L, \text{rig}}^\dagger[\ell_T]$. So the two actions coincides as in the proof of Corollary 6.13. \square

8. Finite Difference equations

In this section we apply the previous theory to the so called finite difference equations, i.e. to the case in which the automorphism $\sigma := \sigma_{q,h}$ is given by $\sigma_{q,h}(f(T)) := f(qT+h)$, with $(q, h) \in K^\times \times K$. This section generalizes the papers [ADV04] and [Pul07] in which one consider p -adic q -difference equations (i.e. of the type $\sigma := \sigma_{q,0}$, with $h=0$). Notice that in this section K is an arbitrary (not necessarily p -adic) ultrametric complete valued field of characteristic 0.

8.1 Infinitesimality of $\sigma_{q,h}$.

The infinitesimality condition for $\sigma_{q,h}$ is $|(q-1)T+h|_* < \rho_{|\cdot|_*, X}$, for all $|\cdot|_* \in \mathcal{M}(X)$. Thank to Lemma 5.3 we are reduced to check this condition on the maximal Skeleton \mathcal{S}_X of $\mathcal{M}(X)$ (cf. section 2.5). We find the following

LEMMA 8.1 (infinitesimality of $\sigma_{q,h}$). *Let $X = D^+(c_0, R_0) - \cup_{i=1, \dots, n} D^-(c_i, R_i)$ be an affinoid. Let $(q, h) \in K^\times \times K$. The automorphism $\sigma_{q,h}$ is infinitesimal on $\mathcal{H}_K(X)$ if and only if*

$$|q-1| < 1, \quad \text{and} \quad |(q-1)c_i + h| < R_i, \quad \text{for all } i = 0, \dots, n. \tag{8.1}$$

Proof. The elements of the maximal Skeleton are those of the form $|\cdot|_{c_i, \rho}$, with $R_i \leq \rho \leq R_0$. The condition on these elements is $|(q-1)T + h|_{c_i, \rho} < \rho$ for all $R_i \leq \rho \leq R_0$. One has $|(q-1)T + h|_{c_i, \rho} = |(q-1)(T - c_i) + (q-1)c_i + h|_{c_i, \rho} = \max(|q-1|\rho, |(q-1)c_i + h|)$. \square

8.2 Non degeneracy of $\sigma_{q,h}$

Let $X = D^+(c_0, R_0) \cup_{i=1, \dots, n} D^-(c_i, R_i)$ be an affinoid, and let $(q, h) \in K^\times \times K$ be a pair satisfying $|q-1| < 1$ and $|(q-1)c_i + h| < R_i$ for all $i = 1, \dots, n$ (cf. Lemma 8.1) in order that $\sigma_{q,h}$ acts infinitesimally on X . Non degeneracy of $\sigma_{q,h}$ will follow from the existence of a (q, h) -Taylor expansion formula for analytic functions on (sufficiently large) discs (cf. Proposition 8.3 below). For this we need some definitions.

For all natural number $n \geq 0$ we set $[n]_q := 1 + q + q^2 + \dots + q^{n-1}$, and $[n]_q^! := [1]_q \cdot [2]_q \cdot [3]_q \cdot \dots \cdot [n]_q$. For all complete valued extensions Ω/K , denote by $\mathfrak{s}_{q,h} : \Omega \rightarrow \Omega$ the map $\mathfrak{s}_{q,h}(x) := qx + h$. For all $c \in \Omega$, and all $n \geq 0$, we set

$$(T - c)_{q,h}^{[n]} := (T - c)(T - \mathfrak{s}_{q,c}(c))(T - \mathfrak{s}_{q,c}^2(c)) \dots (T - \mathfrak{s}_{q,c}^{n-1}(c)). \quad (8.2)$$

We define the Twisted q -binomial by $\binom{n}{i}_q$ by the relation $(1 - T)(1 - qT) \dots (1 - q^{n-1}T) = \sum_{i=0}^n (-1)^i \binom{n}{i}_q \cdot q^{\frac{i(i-1)}{2}} T^i$ in which it is understood that, for $i = 0$, the symbol $q^{\frac{i(i-1)}{2}}$ is equal to 1. It follows from the definition that for $1 \leq i \leq n-1$ one has $\binom{n}{i}_q = \binom{n-1}{i-1}_q + q^i \binom{n-1}{i}_q = q^{n-i} \binom{n-1}{i-1}_q + \binom{n-i}{i}_q$. We call *twisted (q, h) -derivation* the operator

$$\Delta_{q,h} := \frac{\sigma_{q,h} - \text{Id}}{\sigma_{q,h}(T) - T} = \frac{\sigma_{q,h} - \text{Id}}{(q-1)T + h}. \quad (8.3)$$

One has the relations $\sigma_{q,h} \circ \Delta_{q,h} = q\Delta_{q,h} \circ \sigma_{q,h}$, and $\Delta_{q,h}(fg) = \sigma_{q,h}(f)\Delta_{q,h}(g) + \Delta_{q,h}(f)g$, and more generally for all $q \in K^\times$ one has $\Delta_{q,h}^n(fg)(T) = \sum_{i=0}^n \binom{n}{i}_q \cdot \Delta_{q,h}^{n-i}(f)(\sigma_{q,h}^i(T)) \cdot \Delta_{q,h}^i(g)(T)$. The family $\{(T - c)_{q,h}^{[n]}\}_{n \geq 0}$ is the (q, h) -deformation of the family $\{(T - c)^n\}_{n \geq 0}$, in the sense that it is adapted to the (q, h) -derivation. Indeed, for all $n \geq 1$, one has

$$\Delta_{q,h}((T - c)_{q,h}^{[n]}) = [n]_q \cdot (T - c)_{q,h}^{[n-1]}. \quad (8.4)$$

REMARK 8.2. Notice that all the above properties hold even if q is a root of unity. We remark that, if q is a m -th root of unity different from 1, then one has $[n]_q^! = 0$ for all $n \geq m$. For $q = 1$ we have $[n]_q^! = n!$, for all $n \geq 0$.

PROPOSITION 8.3. *Let Ω/K be a complete valued field extension, let $c \in \Omega$, and let $R > 0$ be a real number. Assume that $|q-1| < 1$, and that $|(q-1)c + h| < R$, in order that the disc $D^-(c, R)$ is invariant under the action of $\sigma_{q,h}$. Let $f(T) := \sum_{n \geq 0} a_n (T - c)^n \in \mathcal{A}_\Omega(c, R)$ be an analytic function on $D^-(c, R)$. Then*

- i) $f(T)$ can be uniquely written as $f(T) = \sum_{n \geq 0} \tilde{a}_n (T - c)_{q,h}^{[n]} \in \mathcal{A}_\Omega(c, R)$;
- ii) For all ρ satisfying $|(q-1)c + h| < \rho < R$ one has

$$|f|_{c, \rho} := \sup_{n \geq 0} |a_n| \rho^n = \sup_{n \geq 0} |\tilde{a}_n| \rho^n \quad (8.5)$$

- iii) The radius of convergence of f is given by the formula:

$$\text{Rad}(f, c) := \liminf_n |a_n|^{-1/n} = \liminf_n |\tilde{a}_n|^{-1/n}. \quad (8.6)$$

- iv) Under the above conditions assume moreover that one of the following conditions is fulfilled:

- (a) q is not a root of unity,
- (b) $q = 1$ but $h \neq 0$.

Then one has the (q, h) -Taylor expansion formula

$$f(T) = \sum_{n \geq 0} \Delta_{q,h}^n(f)(c) \cdot \frac{(T-c)_{q,h}^{[n]}}{[n]_q!}. \quad (8.7)$$

Proof. The proof follows closely [Pul07, Lemma 5.3], we omit it for expository reasons. \square

COROLLARY 8.4. *Under the assumptions of point iv) of Proposition 8.3 for all ρ satisfying $|(q-1)c+h| < \rho < R$, one has $\|\Delta_{q,h}\|_{c,\rho} = \rho^{-1}$. where $\|\Delta_{q,h}\|_{c,\rho} := \sup_{f \in \mathcal{A}_\Omega(c,R)} \frac{|\Delta_{q,h}(f)|_{c,\rho}}{|f|_{c,\rho}}$.*

Proof. By the twisted (q, h) -Taylor formula we deduce $|\Delta_{q,h}(f)|_{c,\rho} \leq \rho^{-1}|f|_{c,\rho}$. Now since $|\Delta_{q,h}(T-c)|_{c,\rho} = 1 = \rho^{-1}|T-c|_{c,\rho}$ one has the required equality. \square

COROLLARY 8.5. *Let $(q, h) \in K^\times \times K$. Let Ω/K be a complete valued field extension, and let $c \in \Omega$. Assume that $|q-1| < 1$, and that $|(q-1)c+h| < R$ in order that $\sigma_{q,h}$ acts on $\mathcal{A}_\Omega(c, R)$. If $(q, h) \neq (1, 0)$, then the (q, h) -derivation $\Delta_{q,h}$ acts as well on $\mathcal{A}_\Omega(c, R)$. Assume now that $\sigma_{q,h}$ acts infinitesimally on $\mathcal{H}_K(X)$, then $\Delta_{q,h}$ operates as well on $\mathcal{H}_K(X)$.*

Proof. The operator $\Delta_{q,h}$ has a denominator. If the zero $-h/(q-1)$ of the denominator does not belong to the disc $D^-(c, R)$, or if $q = 1$, then the denominator is invertible in $\mathcal{A}_\Omega(c, R)$, and there is nothing to prove. Assume then that $q \neq 1$, and that $-h/(q-1) \in D^-(c, R)$. Every function in $\mathcal{A}_\Omega(c, R)$ can be written as $f(T) = \sum_{n \geq 0} \tilde{a}_n (T-c)_{q,h}^{[n]}$. Without loss of generality we can assume that $c = -h/(q-1)$, and in this case one sees that $\sigma_{q,h}(f) - f$ is divisible by $(q-1)T + h$.

The case of $\mathcal{H}_K(X)$ is proved as follows. If $-h/(q-1)$ does not belong to $X(K)$, or if $q = 1$, then the denominator is invertible and there is nothing to prove. If $-h/(q-1)$ belongs to $X(K)$, by the above argument one sees that for all $f \in \mathcal{H}_K(X)$, $\Delta_{q,h}(f)$ have no poles on X . \square

COROLLARY 8.6 (Non degeneracy of $\sigma_{q,h}$). *Let $X := D^+(c_0, R_0) - \cup_{i=1,\dots,n} D^-(c_i, R_i)$ be an affinoid. Assume that $\sigma_{q,h}$ is infinitesimal (cf. Lemma 8.1). Then $\sigma_{q,h}$ is non degenerate if and only if (q, h) verifies the conditions iv) of Proposition 8.3. Moreover in this case every point of $X(\Omega)$ is a base point for $\sigma_{q,h}$ (Ω/K being as usual an arbitrary complete valued field extension).*

Proof. Let Ω/K be a complete valued field extension, let $c \in X(\Omega)$. Since $\sigma_{q,h}$ is infinitesimal, the disc $D^-(c, \rho_{c,X})$ is invariant under the action of $\sigma_{q,h}$. By Proposition 8.3 a function $f \in \mathcal{A}_\Omega(c, \rho_{c,X})$ can be uniquely written as $f = \sum_{n \geq 0} \Delta_{q,h}^n(f)(c) (T-c)_{q,h}^{[n]} / [n]_q!$. Clearly $\sigma_{q,h}(f) = f$ if and only if $\Delta_{q,h}(f) = 0$. By the (q, h) -Taylor expansion, this implies that f is constant. Reciprocally, if (q, h) does not satisfy the conditions iv) of Proposition 8.3, then the function $(T-c)_{q,h}^{[n]}$ is fixed by $\sigma_{q,h}$, for $n \gg 0$, because $[n]_q! = 0$ (cf. relation (8.4)). \square

COROLLARY 8.7. *Theorem 5.18 holds without changes for $\Sigma = \{\sigma_{q,h}\}$, where the conditions of infinitesimality and non degeneracy of $\sigma_{q,h}$ have been expressed in Lemma 8.1 and Corollary 8.6. \square*

8.2.1 Roots of unity. Let B_K denotes one of the rings \mathcal{R}_K and \mathcal{H}_K^\dagger . Denote by $B_{K_\infty} := \cup_{n \geq 0} B_{K_n}$, where $K_n = K(\mu_{p^n})$, and define $d - \text{Mod}(B_{K_\infty})$ as the inductive limit of $d - \text{Mod}(B_{K_n})$. The group μ_{p^∞} acts on B_{K_∞} by $\sigma_\xi(f(T)) := f(\xi T)$, $\xi \in \mu_{p^\infty}$. The action of μ_{p^∞} on \mathcal{H}_K^\dagger is infinitesimal and non degenerate since $\mathcal{A}_{K_\infty}(1, 1)^{\mu_{p^\infty}} = K_\infty$.

COROLLARY 8.8. *The deformation functor provides equivalences*

$$d - \text{Mod}(B_{K_\infty})^{[1]} \xrightarrow{\sim} \mu_{p^\infty} - \text{Mod}(B_{K_\infty})^{\text{adm}}, \quad (8.8)$$

$$d - \text{Mod}(\mathcal{R}_{K_\infty})^{(\varphi)} \xrightarrow{\sim} \mu_{p^\infty} - \text{Mod}(\mathcal{R}_{K_\infty})^{(\varphi), \text{adm}}. \quad (8.9)$$

If $\mathcal{I}_{k^{\text{alg}}((t))}$ denotes the inertia of $\text{Gal}(k((t))^{\text{alg}}/k((t)))$, then one has an equivalence

$$\text{Rep}_{K^{\text{alg}}}(\mathcal{I}_{k^{\text{alg}}((t))}) \xrightarrow{\sim} \mu_p - \text{Mod}(\mathcal{R}_{K_\infty})^{(\varphi), \text{adm}}. \quad (8.10)$$

Proof. The first assertion for $B_{K_\infty} := \mathcal{R}_{K_\infty}$ follows from the case $B_{K_\infty} := \mathcal{H}_{K_\infty}^\dagger$ using the canonical extension functor. The result over $\mathcal{H}_{K_\infty}^\dagger$ follows by Theorem 5.22. Now the second assertion follows from [And02] which provides an equivalence between $\text{Rep}_{K^{\text{alg}}}(\mathcal{I}_{k^{\text{alg}}((t))})$ and $d - \text{Mod}(\mathcal{R}_{K_\infty})^{(\varphi)}$. \square

8.3 Admissible $\sigma_{q,h}$ -modules

In this subsection we describe the categories $d - \text{Mod}(\mathcal{H}_K(X))^{\text{adm}(\sigma_{q,h})}$ and $\sigma_{q,h} - \text{Mod}(\mathcal{H}_K(X))^{\text{adm}}$. We give conditions to provide the $\sigma_{q,h}$ -compatibility of a $\sigma_{q,h}$ -module. In the particular case of the operator of the form $\sigma_{q,h}$ the analytic cocycle attached to an admissible object can be described explicitly using the (q, h) -Taylor expansion. Assume that $\sigma_{q,h}$ acts infinitesimally on X .

Let $(M, \sigma_{q,h}^M)$ be a (not necessarily Taylor admissible) $\sigma_{q,h}$ -module, and let $\mathbf{e} \subset M$ be a fixed basis of M . Let $\sigma_{q,h}(Y) = A \cdot Y$ is the (q, h) -difference equation defined by $\sigma_{q,h}^M$ in the basis \mathbf{e} , and let $\Delta_{q,h}(Y) = G(q, h; T) \cdot Y$, where $G(q, h; T) := \frac{A-1}{\delta_{\sigma_{q,h}}(T)}$ be the corresponding equation in term of the twisted (q, h) -derivation.

Assume, for a moment, that M is Taylor admissible, then, by the twisted (q, h) -Taylor formula, the analytic cocycle attached to the equation is given by the expression

$$Y(x, y) := \sum_{n \geq 0} G_{[n]}(q, h; T)(y) \frac{(x - y)_{q,h}^{[n]}}{[n]_q!}, \quad (8.11)$$

where $G_{[n]} := G_{[n]}(q, h; T)$ is defined by the relation $\Delta_{q,h}^n(Y) = G_{[n]}Y$, and is given inductively by the expression $G_{[0]} = \text{Id}$, $G_{[1]} = G$, and $G_{[n+1]} = \sigma_{q,h}(G_{[n]})G_{[1]} + \Delta_{q,h}(G_{[n]})$.

Conversely we are now interested to understand, without assuming that M Taylor admissible, when such a formal expression define an analytic cocycle. In this case we define the *formal radius* of M in the basis \mathbf{e} as

$$\text{Rad}_F(M, \mathbf{e}, |\cdot|_*) = \min\left(\liminf_n \left(|G_{[n]}(q, h; T)|_* / |[n]_q!\right)^{-1/n}, \rho_{|\cdot|_*, X}\right). \quad (8.12)$$

A priori the formal Radius depends on the coosen basis \mathbf{e} , and moreover none of the properties of the radius of an analytic cocycle are fulfilled by $\text{Rad}_F(M, \mathbf{e}, |\cdot|_*)$, because the expression (8.11) is formal, and does not necessarily represent an analytic cocycle. The reason for which we consider the formal radius is that it exists on the whole $\mathcal{M}(X)$, while the usual radius exists whenever the expression (8.11) represents a $\sigma_{q,h}$ -compatible analytic cocycle. The following Proposition describe the relations between the formal radius and the usual one.

PROPOSITION 8.9. *Assume that $\sigma_{q,h}$ satisfies condition iv) of Proposition 8.3 in order that it acts infinitesimally on X and that it is non degenerate. The following conditions are equivalent :*

- i) *The expression (8.11) defines a $\sigma_{q,h}$ -compatible cocycle;*
- ii) *For all $|\cdot|_* \in \mathcal{M}(X)$ one has :*

$$|\delta_{\sigma_{q,h}}|_* < \text{Rad}_F(M, \mathbf{e}, |\cdot|_*). \quad (8.13)$$

- iii) *Condition (8.13) holds for the elements of the maximal Skeleton \mathcal{S}_X of X ;*
- iv) *Condition (8.13) holds for the critical elements \mathcal{P}_X of the maximal Skeleton of X .*

Proof. If $Y(x, y)$ defines a $\sigma_{q,h}$ -compatible cocycle, then $\text{Rad}(Y(x, y), |\cdot|_*) = \text{Rad}_F(M, \mathbf{e}, |\cdot|_*)$, and the properties ii), iii), iv) hold (and are equivalents by Propositions 5.5). Assume now that condition

ii) holds, then an estimation shows that the series of functions (8.11) converges and that defines an analytic cocycle which is automatically $\sigma_{q,h}$ -compatible. Hence i) holds.

Now assume that iii) holds. As in the proof of Proposition 5.5, if $|\cdot|_* \leq |\cdot|_{**}$ then $Rad_F(M, \mathbf{e}, |\cdot|_*) \geq Rad_F(M, \mathbf{e}, |\cdot|_{**})$ and $\rho_{|\cdot|_*, X} = \rho_{|\cdot|_{**}, X}$, then it is sufficient to prove the condition on an exhaustive family, in particular on the maximal Skeleton of X . This proves that iii) implies ii).

Now assume that the condition iv) holds, we want to prove that iii) holds as well without assuming that $Y(x, y)$ is an analytic cocycle. In particular we can not use the continuity of the function $|\cdot|_* \mapsto Rad_F(M, \mathbf{e}, |\cdot|_*)$ on the whole $\mathcal{M}(X)$ as we did in Proposition 5.5. We proceed then as follows. With the notations of Proposition 2.13 we consider the restriction of the function $\rho \mapsto Rad_F(M, \mathbf{e}, |\cdot|_{c_i, \rho})$ to the interval $[\rho_{i,s}, \rho_{i,s+1}]$. This function is the liminf of the log-concave functions $(|G_{[n]}|_{c_i, \rho} / |n|_q!)^{-1/n}$, for this reason $\rho \mapsto Rad_F(M, \mathbf{e}, |\cdot|_{c_i, \rho})$ is log-concave on $[\rho_{i,s}, \rho_{i,s+1}]$ and hence continuous on the open interval $]\rho_{i,s}, \rho_{i,s+1}[$. The log-concavity on $[\rho_{i,s}, \rho_{i,s+1}]$ implies that the condition (8.13) is verified for all $\rho \in [\rho_{i,s}, \rho_{i,s+1}]$ if and only if it is verified at $\rho_{i,s}$ and $\rho_{i,s+1}$, because the log-concavity implies $Rad_F(M, \mathbf{e}, |\cdot|_{c_i, \rho_{i,s}}) \leq \lim_{\rho \rightarrow \rho_{i,s}^-} Rad_F(M, \mathbf{e}, |\cdot|_{c_i, \rho})$ and $Rad_F(M, \mathbf{e}, |\cdot|_{c_i, \rho_{i,s+1}}) \leq \lim_{\rho \rightarrow \rho_{i,s+1}^+} Rad_F(M, \mathbf{e}, |\cdot|_{c_i, \rho})$. This proves that iv) implies iii). \square

COROLLARY 8.10. *Let $X = D^+(c_0, R_0) - \cup_{i=1}^n D^-(c_i, R_i)$. The category $d - \text{Mod}(\mathcal{H}_K(X))^{\text{adm}(\sigma_{q,h})}$ (resp. $\sigma_{q,h} - \text{Mod}(\mathcal{H}_K(X))^{\text{adm}}$) is the full subcategory of $d - \text{Mod}(\mathcal{H}_K(X))$ (resp. $\sigma_{q,h} - \text{Mod}(\mathcal{H}_K(X))$) whose objects M satisfy*

$$\begin{cases} \max(|q-1||c_i - c_j|, |(q-1)c_i + h|) < Rad(M, |\cdot|_{c_i, |c_i - c_j|}), & \text{for all } i, j = 1, \dots, n \\ \max(|q-1|R_i, |(q-1)c_i + h|) < Rad(M, |\cdot|_{c_i, R_i}) & \text{for all } i = 0, \dots, n \end{cases} \quad (8.14)$$

(resp. the same condition with $Rad_F(M, \mathbf{e}, -)$). \square

8.4 Germs of admissible \mathcal{Q} -actions

Analogously to section 7, also in this context we have a group action. The group of operator is the following. We consider a new group law on the set $\mathcal{Q} := K^\times \times K$ given by

$$(q_1, h_1) \cdot (q_2, h_2) := (q_1 q_2, q_2 h_1 + h_2). \quad (8.15)$$

The identity of \mathcal{Q} is $(1, 0)$, the inverse of (q, h) is $(q^{-1}, -q^{-1}h)$, and $(q, h)^n = (q^n, [n]_q \cdot h)$. Moreover $(q, h) = (q, 0) \cdot (1, h)$ for all $(q, h) \in \mathcal{Q}$, and the action of the conjugation is $(q, h) \cdot (\bar{q}, \bar{h}) \cdot (q, h)^{-1} = (\bar{q}, q^{-1}((\bar{q} - 1)h + \bar{h}))$. This shows that the sub-group $\mathfrak{H} := \{(q, h) \mid q = 1\}$ is normal in \mathcal{Q} , and hence \mathcal{Q} is the semidirect product of \mathfrak{H} and $\mathfrak{Q} := \{(q, h) \mid h = 0\}$.

We consider \mathcal{Q} as a topological group with the direct product topology of $K^\times \times K$ in which each factor has the topology induced by the absolute value.

REMARK 8.11. Let M be a $\mathcal{Q}_{\tau, \nu}$ -module. Let $A(q, h; T)$ be the matrix of $\sigma_{q, h}^M$ is a given basis. Then

$$A((q, h) \cdot (q', h'); T) = A(q', h'; qT + h) \cdot A(q, h; T). \quad (8.16)$$

In particular one has

$$A((q, h)^n; T) = A(q, h; q^{n-1}T + [n-1]_q h) \cdot A(q, h; q^{n-2}T + [n-2]_q h) \cdots A(q, h; T). \quad (8.17)$$

8.4.1 Germs of admissible \mathcal{Q} -actions. As in section 7 we define now the category of germs of admissible \mathcal{Q} -actions. For all (τ, ν) , with $\nu > 0$, $0 < \tau \leq 1$, we set

$$\mathcal{Q}_{\tau, \nu} := D_K^-(1, \tau) \times D_K^-(0, \nu). \quad (8.18)$$

$\mathcal{Q}_{\tau, \nu}$ is an open subgroup of \mathcal{Q} , and $\{\mathcal{Q}_{\tau, \nu}\}_{\tau, \nu > 0}$ is a basis of neighborhoods of the identity in \mathcal{Q} .

Let X be an affinoid. A *germ of admissible \mathcal{Q} -action* is a finite free $\mathcal{H}_K(X)$ -module, together with a *Taylor admissible* semilinear action of an unspecified sub-group $\mathcal{Q}_{\tau,\nu}$. Morphisms between admissible germs of \mathcal{Q} -actions are $\mathcal{H}_K(X)$ -linear maps commuting with the action of an unspecified $\mathcal{Q}_{\tau,\nu}$. We denote by $\text{Germ}(\mathcal{Q})^{\text{adm}}$ the category of admissible germs of \mathcal{Q} -actions. For fixed (τ, ν) one has the following description of the category of $\mathcal{Q}_{\tau,\nu}$ -admissible $\mathcal{H}_K(X)$ -differential modules.

LEMMA 8.12. *Let $\max(\tau, \nu) > 0$. Assume that $\mathcal{Q}_{\tau,\nu}$ acts infinitesimally on X . An analytic cocycle $Y(x, y)$ is $\mathcal{Q}_{\tau,\nu}$ -compatible if and only if*

$$\begin{cases} \max(\tau|c_i - c_j|, \tau|c_i|, \nu) & \leq & \text{Rad}(Y(x, y), |\cdot|_{c_i, |c_i - c_j|}), & \text{for all } i, j = 1, \dots, n \\ \max(\tau R_i, \tau|c_i|, \nu) & \leq & \text{Rad}(Y(x, y), |\cdot|_{c_i, R_i}) & \text{for all } i = 0, \dots, n \end{cases} \quad (8.19)$$

Reciprocally a $\mathcal{Q}_{\tau,\nu}$ -module M is Taylor admissible if and only if the same conditions holds for $\text{Rad}_F(M, \mathbf{e}, -)$.

COROLLARY 8.13. *The category $d - \text{Mod}(\mathcal{H}_K(X))$ is equivalent to $\text{Germ}(\mathcal{Q})^{\text{adm}}$.* \square

8.5 Effective computations

If we know the matrix $A(q, h; T)$ of the action of $\sigma_{q,h}^M : M \rightarrow M$ in a basis, we would like to have an algorithm to know as well the attached differential equation. The algorithm should not involve the cocycle which is usually unknown. For this we have to introduce the Lie algebra of \mathcal{Q} .

8.5.1 *The Lie algebra $\mathcal{L}(\mathcal{Q})$.* We define $\mathcal{L} := \mathcal{L}(\mathcal{Q})$ as the K -vector space of analytic paths $\gamma : D^-(0, \varepsilon) \rightarrow \mathcal{Q}$, for some ε (depending on γ), truncated at the first order. In other words a path γ of \mathcal{L} is of the form $\gamma_{(a,b)}(r) := (1 + ar, br)$, for some $(a, b) \in K^2$. The multiplications in \mathcal{Q} gives $\gamma_{(a,b)}(r) \cdot \gamma_{(a',b')}(r) = (1 + (a + b')r + abr^2, (b + b')r + abr^2)$, which truncated at the first order gives the rule $\gamma_{(a,b)} \cdot \gamma_{(a',b')} = \gamma_{(a+a', b+b')}$. This gives an isomorphism of K -vector spaces $\gamma_{(a,b)} \mapsto (a, b) : \mathcal{L}(\mathcal{Q}) \xrightarrow{\sim} K^2$. The conjugated of a path is again a path and one has $(q, h) \cdot \gamma_{(a,b)}(r) = \gamma_{(a, (ah+b)q^{-1})}(r) \cdot (q, h)$.

REMARK 8.14. Let B be one of the rings $\mathcal{H}_K(X)$, $\mathcal{H}_K^\dagger(X)$, $\mathcal{A}_K(I)$, \mathcal{R}_K . The limit “ $\frac{d}{dr}(\sigma_{\gamma_{(a,b)}(r)})$ ” := $\lim_{r \rightarrow 0} \frac{\sigma_{1+ar, br} - \text{id}}{r}$ converges to the derivation $(aT + b)d/dT$ of B , with respect to the simple convergence topology of $\text{End}_K^{\text{cont}}(B)$. Hence we have an injective K -linear map $d : \mathcal{L} \rightarrow \text{Der}_K(B)$, sending (a, b) into $d_{(a,b)} := (aT + b)d/dT$. One has $\sigma_{q,h} \circ d_{(a,b)} = d_{(a, \frac{ah+b}{q})} \circ \sigma_{q,h}$. In particular $d_{(1,0)} = Td/dT$ and $d_{(0,1)} = d/dT$.

8.5.2 *Algorithm to compute the connection from the action of $\mathcal{Q}_{\tau,\nu}$.* Let $(M, \mathcal{Q}_{\tau,\nu}^M)$ be a germ of admissible \mathcal{Q} -action. Assume that the action of $\mathcal{Q}_{\tau,\nu}$ is known. Let $A(q, h; T)$ be the matrix of $\sigma_{q,h}^M$ in a fixed basis. We notice that the map $(q, h; T) \mapsto A(q, h; T)$ is analytic on $\mathcal{Q}_{\tau,\nu} \times X$ because of the analyticity of the cocycle $Y(x, y)$. Indeed one has $A(q, h; T) = Y(qT + h, T)$.

PROPOSITION 8.15. *Let $Y' = GY$ be the differential equation attached to $(M, \mathcal{Q}_{\tau,\nu}^M)$, then for all $(a, b) \in K^2 - \{(0, 0)\}$ one has*

$$G(T) = (aT + b)^{-1} \cdot \left[a \cdot \frac{\partial}{\partial q} A(q, h; T) \Big|_{q=1, h=0} + b \cdot \frac{\partial}{\partial h} A(q, h; T) \Big|_{q=1, h=0} \right]. \quad (8.20)$$

In particular $G(T) = T^{-1} \left[\frac{d}{dq} A(q, h; T) \right]_{q=1, h=0} = \left[\frac{d}{dh} A(q, h; T) \right]_{q=1, h=0}$.

Proof. To prove this equality in $\mathcal{H}_K(X)$ it is enough to prove it in the larger ring $\mathcal{A}_\Omega(y_0, R)$, for $y_0 \in X(\Omega)$, and $R = \text{Rad}(Y(x, y), y_0)$, Ω/K being as usual a complete valued field extension. Then one can write $A(q, h; T) = Y(qT + h, y_0)Y(T, y_0)^{-1}$, and $G(T) = d/dT(Y(T, y_0))Y(T, y_0)^{-1}$. Now

the expression (8.20) equals the following

$$(aT + b)^{-1} \cdot \frac{d}{dr} \left(A(1 + ar, br; T) \right) \Big|_{r=0} = (aT + b)^{-1} \lim_{r \rightarrow 0} \frac{A(1 + ar, br; T) - \text{Id}}{r}. \quad (8.21)$$

This last divided by $(aT + b)$ is equal to

$$\lim_{r \rightarrow 0} \frac{Y((1 + ar)T + br, y_0) - Y(T, y_0)}{r} \cdot Y(T, y_0)^{-1} = d_{(a,b)}(Y(T, y_0))Y(T, y_0)^{-1}, \quad (8.22)$$

which coincides with $(aT + b) \cdot d/dT(Y(T, y_0)) \cdot Y(T, y_0)^{-1} = (aT + b) \cdot G(T)$. \square

8.5.3 Algorithm to compute the connection from the action of an individual $\sigma_{q,h}$. Let $(M, \sigma_{q,h}^M)$ be an admissible $\sigma_{q,h}$ -module. Let $A(q, h; T)$ be the matrix of $\sigma_{q,h}^M$ in a fixed basis. Then

COROLLARY 8.16. *Let $Y' = GY$ be the differential equation attached to $(M, \sigma_{q,h}^M)$, then one has*

$$G(T) = \lim_{n \rightarrow +\infty} \frac{A(q^{p^n}, [p^n]_q h; T) - I}{(q^{p^n} - 1)T + [p^n]_q h}. \quad (8.23)$$

Proof. This follows as in Proposition 8.15 from the fact that $\lim_{\sigma \rightarrow \text{Id}} \frac{\sigma - \text{Id}}{\sigma(T) - T} = d/dT$, and from that $\lim_{n \rightarrow \infty} (q, h)^{p^n} = \lim_{n \rightarrow \infty} (q^{p^n}, [p^n]_q h) = (1, 0)$, hence $\lim_{n \rightarrow \infty} \sigma_{q^{p^n}, [p^n]_q h} = \text{Id}$. \square

8.6 $\sigma_{q,h}$ -modules over the Robba ring.

The infinitesimality condition for the operator $\sigma_{q,h}$ over the Robba ring is given by

$$\max(|q - 1|, |h|) < 1. \quad (8.24)$$

The non degeneracy condition is always given by condition iv) of Proposition 8.3. The compatibility condition is given by $\max(|q - 1| \cdot \rho, |h|) < \text{Rad}(Y(x, y), \rho)$, for all $\rho < 1$ sufficiently close to 1 (resp. the same condition for $\text{Rad}_F(M, \mathbf{e}, \rho)$). Notice that if $\sigma_{q,h}$ is infinitesimal, then every solvable differential module is automatically σ -compatible.

COROLLARY 8.17. *Theorem 6.1 holds without changes for $\Sigma = \{\sigma_{q,h}\}$, where the conditions of infinitesimality and non degeneracy of $\sigma_{q,h}$ have been expressed explicitly in equations (8.24) and Corollary 8.6.* \square

COROLLARY 8.18. *If $\sigma_{q,h}$ is infinitesimal and non degenerate, then every Taylor admissible $\sigma_{q,h}$ -modules admitting a Frobenius structure over \mathcal{R}_K is quasi unipotent.*

COROLLARY 8.19. *Assume that $\sigma_{q,h}$ is infinitesimal. The category $d - \text{Mod}(\mathcal{R}_K(X))^{\text{adm}(\sigma_{q,h})}$ (resp. $\sigma_{q,h} - \text{Mod}(\mathcal{R}_K(X))^{\text{adm}}$) is the full subcategory of $d - \text{Mod}(\mathcal{R}_K(X))$ (resp. $\sigma_{q,h} - \text{Mod}(\mathcal{R}_K(X))$) whose objects M satisfy*

$$\max(|q - 1| \cdot \rho, |h|) < \text{Rad}(M, \rho), \quad (8.25)$$

for all ρ in some I_ε . (resp. the same condition with Rad_F). In particular solvable objects are automatically $\sigma_{q,h}$ -compatible. \square

LEMMA 8.20. *Let $\tau, \nu > 0$. An analytic cocycle $Y(x, y)$ over \mathcal{R}_K is $\mathcal{Q}_{\tau, \nu}$ -compatible if and only if*

$$\max(\tau \rho, \nu) \leq \text{Rad}(Y(x, y), \rho). \quad \square \quad (8.26)$$

COROLLARY 8.21. *The category $\mathcal{Q}_{1,1} - \text{Mod}(\mathcal{R}_K)^{\text{adm}}$ is equivalent to $d - \text{Mod}(\mathcal{R}_K)^{\text{adm}(\mathcal{Q}_{1,1})} = d - \text{Mod}(\mathcal{R}_K)^{\text{sol}}$ of solvable differential equations.* \square

9. Morita's p -adic Gamma function and Kubota-Leopoldt's L -functions

In this section we apply the previous theory to a particular difference equation satisfied by the Morita's p -adic Γ -function. As a consequence we relate the radius of convergence of the associated differential equation to the absolute value of some value at *positive integers* of the L functions appearing in the Taylor expansion of the function $\log(\Gamma)$. We prove then, as a direct consequence of the deformation, a family of congruences between these values.

9.1 A Small radius Lemma for rank one (q, h) -difference equations

We give here a Lemma which expresses the radius of convergence of a rank one (q, h) -difference equation directly in terms of the matrix of the derivation, under the assumption that the radius is “small”, or equivalently that the norm of the matrix of the connection (resp. of the (q, h) -derivation) is “big”.

In this sub-section we assume that $(q, h) \in K^2$ satisfies the conditions of Lemma 8.1 and Corollary 8.6, in order that $\sigma_{q,h}$ is infinitesimal and non degenerate.

For all $q \in K^\times$, $|q - 1| < 1$, we set $\omega_q := \lim_{n \rightarrow \infty} |[n]_q|^{1/n}$. One verifies (cf. [DV04, 3.5]) that, if κ is the smallest positive integer such that $|q^\kappa - 1| < \omega$, then

$$\omega_q = \begin{cases} \omega & \text{if } \kappa=1 \\ ([\kappa]_q \cdot \omega)^{1/\kappa} & \text{if } \kappa \geq 2 \end{cases} \quad (9.1)$$

Let $|\cdot|_* \in \mathcal{M}(X)$, and let $\Delta_{q,h} := \frac{\sigma_{q,h}-1}{(q-1)T+h}$. We set

$$\|\Delta_{q,h}\|_* := \sup_{f \in \mathcal{H}_K(X)} \frac{|\Delta_{q,h}(f)|_*}{|f|_*}, \quad \|d/dx\|_* := \sup_{f \in \mathcal{H}_K(X)} \frac{|d/dx(f)|_*}{|f|_*}. \quad (9.2)$$

LEMMA 9.1. *Let $X = D^+(c_0, R_0) - \cup_{i=1}^n D^-(c_i, R_i)$, and let $|\cdot|_{y,\rho}$ be a multiplicative seminorm of $\mathcal{M}(X)$. Then $\|d/dx\|_{y,\rho} \leq \rho^{-1}$, and if $y \in K$ equality holds. Assume now that $(q, h) \in K^2$ is infinitesimal and non degenerate (cf. Lemma 8.1 and Corollary 8.6). If $|(q-1)y + h| < \rho$, then $\|\Delta_{q,h}\|_{y,\rho} \leq \rho^{-1}$, and if $y \in K$, then equality holds.*

Proof. Let $t \in X(\Omega)$ be a Dwork generic point for $|\cdot|_{y,\rho}$, where Ω/K is a sufficiently large complete valued extension. Then, one has $|t - y|_\Omega = |T - y|_{y,\rho} = \rho$ and hence $|\cdot|_{t,\rho} = |\cdot|_{y,\rho}$ (cf. Remark 2.4). Notice that y does not necessarily belong to X , while $t \in X(\Omega)$. Moreover, the assumptions imply $|(q-1)t + h| = |(q-1)(t-y) + (q-1)y + h| < \rho$, because $|q-1| < 1$. We consider then the inclusion $\mathcal{H}_K(X) \subset \mathcal{A}_\Omega(t, \rho)$. By Corollary 8.4, for all $|(q-1)t + h| < \rho' < \rho$ one has $\|\Delta_{q,h}\|_{t,\rho'} = (\rho')^{-1}$, in particular $\frac{|\Delta_{q,h}(f)|_{y,\rho'}}{|f|_{y,\rho'}} \leq (\rho')^{-1}$. Hence by continuity $\frac{|\Delta_{q,h}(f)|_{y,\rho}}{|f|_{y,\rho}} \leq \rho^{-1}$, for all $f \in \mathcal{H}_K(X)$. Now if $y \in K$, then $(T-y) \in \mathcal{H}_K(X)$ and hence $|\Delta_{q,h}(T-y)|_{y,\rho} = 1 = \rho^{-1}|T-y|_{y,\rho}$, hence we have the equality $\|\Delta_{q,h}\|_{y,\rho} = \rho^{-1}$. An analogous proof applies to d/dx . \square

9.1.1 Let $(M, \sigma_{q,h}^M)$ be a rank one $\sigma_{q,h}$ -difference module (not necessarily Taylor admissible). Denotes by $\Delta_{q,h}^M : M \rightarrow M$ the operator $\Delta_{q,h}^M := \frac{\sigma_{q,h}^M - 1}{(q-1)T+h}$. Let

$$\Delta_{q,h}(Y) = G_{[1]}(q, h; T) \cdot Y, \quad G_{[1]}(q, h; T) \in \mathcal{H}_K(X), \quad (9.3)$$

be the corresponding rank one (q, h) -difference equation in a fixed basis $\mathbf{e} \in M$ (i.e. $\Delta_{q,h}^M(\mathbf{e}) = \mathbf{e} \cdot G_{[1]}(q, h; T)$). Let $\text{Rad}_F(M, \mathbf{e}, -)$ be the formal radius of M (cf. Section 8.3). We recall that the formal radius $\text{Rad}_F(M, \mathbf{e}, -)$ depends on the chosen basis \mathbf{e} , and that it coincides with the radius $\text{Rad}(Y(x, y), -)$ under the assumptions of Proposition 8.9. The reason for which we consider the formal radius is that it exists on the whole $\mathcal{M}(X)$, while the usual radius exists whenever the formal expression (8.11) represents a σ -compatible analytic cocycle $Y(x, y)$.

LEMMA 9.2 (Small Radius). *Let $|\cdot|_* \in \mathcal{M}(X)$ be a fixed seminorm. Assume that $\sigma_{q,h}$ acts infinitesimally and is non degenerate (cf. Lemma 8.1, and Corollary 8.6). Let M be a rank one $\sigma_{q,h}$ -module. In the above notations, let $G_{[n]} := G_{[n]}(q, h; T)$ be the matrix defined by $\Delta_{q,h}^n(Y) = G_{[n]}Y$. In the notations of Section 8.3, one has the followings:*

i) *One has always*

$$\begin{cases} \liminf_n \left(|G_{[n]}|_* / |[n]_q^!| \right)^{-1/n} \geq \frac{\omega_q}{\max(|G_{[1]}|_*, \|\Delta_{q,h}\|_*)} , \\ \text{Rad}_F(M, \mathbf{e}, |\cdot|_*) \geq \min \left(\frac{\omega_q}{\max(|G_{[1]}|_*, \|\Delta_{q,h}\|_*)} , \rho_{|\cdot|_*, X} \right) . \end{cases} \quad (9.4)$$

ii) *One has $|G_{[1]}|_* > \|\Delta_{q,h}\|_*$ if and only if $\liminf_n (|G_{[n]}|_* / |[n]_q^!|)^{-1/n} < \omega_q \cdot \|\Delta_{q,h}\|_*^{-1}$. In particular if $|\cdot|_* = |\cdot|_{y,\rho}$, with $y \in K$ (cf. Lemma 9.1), then $|G_{[1]}|_{y,\rho} > \rho^{-1}$ if and only if $\text{Rad}_F(M, \mathbf{e}, |\cdot|_{y,\rho}) < \omega\rho$.*

iii) *If $|G_{[1]}|_* > \|\Delta_{q,h}\|_*$ then*

$$\begin{cases} \liminf_n \left(|G_{[n]}|_* / |[n]_q^!| \right)^{-1/n} = \frac{\omega_q}{|G_{[1]}|_*} , \\ \text{Rad}_F(M, \mathbf{e}, |\cdot|_*) = \min \left(\frac{\omega_q}{|G_{[1]}|_*} , \rho_{|\cdot|_*, X} \right) . \end{cases} \quad (9.5)$$

The same statements hold for rank one differential equations replacing $\Delta_{q,h}, \omega_q, [n]_q^!, \text{Rad}_F(M, \mathbf{e}, |\cdot|_)$ with $d/dx, \omega, n!, \text{Rad}(Y(x, y), |\cdot|_*)$.*

Proof. By induction on the formula $G_{[n+1]} = \Delta_{q,h}(G_{[n]}) + \sigma_q(G_{[n]})G_{[1]}$ one finds $|G_{[n]}|_* \leq \max(|G_{[n]}|_*, \|\Delta_{q,h}\|_*^n)$, and equality holds if $|G_{[1]}|_* > \|\Delta_{q,h}\|_*$. Since the sequence $|[n]_q^!|^{1/n}$ is convergent to ω_q , one has $\liminf_n (|G_{[n]}|_* / |[n]_q^!|)^{-1/n} = \omega_q \cdot \liminf_n |G_{[n]}|_*^{-1/n}$. This proves point i). If $|G_{[1]}|_* > \|\Delta_{q,h}\|_*$, then the same induction proves that $|G_{[n]}|_* = |G_{[1]}|_*^n$, and hence $\liminf_n (|G_{[n]}|_* / |[n]_q^!|)^{-1/n} = \omega_q / |G_{[1]}|_* < \omega_q / \|\Delta_{q,h}\|_*$. Reciprocally if $\liminf_n (|G_{[n]}|_* / |[n]_q^!|)^{-1/n} < \omega_q / \|\Delta_{q,h}\|_*$, then the point i) shows that $|G_{[1]}|_* > \|\Delta_{q,h}\|_*$. \square

9.2 Morita's p -adic Gamma function as solution of a difference equation

In this section $K = \mathbb{Q}_p$. Assume that $p \neq 2$ is a prime number. Let $\Gamma_p(T)$ be the Morita's p -adic Gamma function, that is the unique continuous function on \mathbb{Z}_p verifying the functional equation

$$\Gamma_p(0) = 1 , \quad \text{and} \quad \Gamma_p(x+1) = \begin{cases} -x\Gamma_p(x) & \text{if } |x|=1 \\ -\Gamma_p(x) & \text{if } |x|<1 , \end{cases} \quad (9.6)$$

whose value on the natural numbers $n \geq 1$ is given by $\Gamma_p(n) = (-1)^n \cdot \prod_{i=1, (i,p)=1}^{n-1} i$.

It have been known since Morita [Mor75] that $\Gamma_p(T)$ is locally analytic with local radius greater than $|p|$. Subsequently Dwork [Dwo82], applying non cohomological methods introduced by D.Barsky [Bar80], was able to compute the exact radius of convergence of $\Gamma_p(T)$ in a neighborhood of the points $0, \dots, p-1$. We denote by

$$\Gamma_p^0(T) = 1 + \sum_{n \geq 1} \gamma_n^0 \cdot T^n , \quad (9.7)$$

the Taylor expansion at $T = 0$ of $\Gamma_p(T)$ (i.e. $\Gamma_p^0(x) = \Gamma_p(x)$ for all $x \in p\mathbb{Z}_p$). The radius of convergence of $\Gamma_p^0(T)$ is exactly equal to $\omega \cdot |p|^{1/p}$. The Taylor development of Γ_p around the integers $i = 1, \dots, p-1$ is clearly equal to $\Gamma_p^i(T) := (-1)^i (T+1)(T+2) \cdots (T+i-1) \Gamma_p^0(T)$, because of the

functional equation, and it has the same radius of convergence of $\Gamma_p^0(T)$. In particular one has

$$\Gamma_p^0(T+p) = A(1,p;T) \cdot \Gamma_p^0(T), \quad (9.8)$$

where $A(1,p;T) = (-1)^{(p)}(T+1)(T+2)\cdots(T+p-1)$. More generally we consider the equations

$$\Gamma_p^0(T+p^n) = A(1,p^n;T) \cdot \Gamma_p^0(T). \quad (9.9)$$

where $A(1,p^n;T)$ is deduced from $A(1,p;T)$ by the recursive formulas (8.17):

$$A(1,p^n;T) = (-1) \cdot \prod_{i=1, (i,p)=1}^{p^n-1} (T+i). \quad (9.10)$$

The function $\Gamma_p^0(T)$ is hence solution of the (q,h) -difference equation (9.8) and of its p^n th iterations (9.9). These equations are defined everywhere since $A(1,p^n;T)$ is a polynomial. But each one of them verifies the assumptions of Theorem 5.27 over a disk $D^-(0,r_n)$, centered at 0.

By Theorem 5.27 the function $\Gamma_p^0(T)$ is then solution of a differential equation

$$\Gamma_p^0(T)' = g_0(T) \cdot \Gamma_p^0(T). \quad (9.11)$$

Denote by $Y_0(x,y)$ the analytic cocycle attached to this differential equation. In this section we are interested to

- i) find the convergence disk of $g_0(T)$,
- ii) describe the functions $\rho \mapsto |g_0(T)|_\rho$,
- iii) describe the functions $\rho \mapsto \text{Rad}(Y_0(x,y), |\cdot|_\rho)$.

The disk $D^-(0,r_n)$ can be defined as the biggest disk over which $Y_0(x,y)$ is σ_{1,p^n} -admissible. Since the σ_{1,p^n} -admissibility condition is $|\delta_{\sigma_{1,p^n}}|_\rho = |p|^n < \text{Rad}(Y_0(x,y), \rho)$, we have $D^-(0,r_n) \subset D^-(0,r_{n+1})$ for all $n \geq 1$. Moreover if $Y_{p^n}(x,y)$ denotes the restriction of $Y_0(x,y)$ to $D^-(0,r_n) \times D^-(0,r_n)$, then $Y_{p^n}(x,y)$ is by Theorem 5.27, solution of the equation $Y_{p^n}(\sigma_{1,p^n}(x), y) = A(1,p^n;x) \cdot Y_{p^n}(x,y)$. We will prove that

$$\cup_{n \geq 1} D^-(0,r_n) = D^-(0,1). \quad (9.12)$$

As a matter of facts $Y_0(x,y)$ will be constructed by gluing the analytic cocycles $Y_{p^n}(x,y)$ defined step by step, over $D^-(0,r_n) \times D^-(0,r_n)$, by the $(1,p^n)$ -th equation $\sigma_{1,p^n}(Y) = A(1,p^n;T) \cdot Y$ by the expression (8.11). The function $\Gamma_p^0(T)$ will be hence equal to $Y_0(T,0)$.

THEOREM 9.3. *The function $\Gamma_p^0(T)$ is the Taylor solution at $T=0$ of a rank one differential equation:*

$$Y' = g_0(T) \cdot Y, \quad (9.13)$$

which satisfies:

- i) The function $g_0(T)$ belongs to $\mathcal{A}_{\mathbb{Q}_p}(0,1) \subset \mathbb{Q}_p[[T]]$;
- ii) If $Y_0(x,y)$ is the analytic cocycle over $D^-(0,1) \times D^-(0,1)$, defined as Taylor solution of the equation (9.13), then one has:

$$\text{Ray}(Y_0(x,y), |\cdot|_\rho) = \begin{cases} \omega|p|^{1/p} & \text{if } 0 \leq \rho \leq \omega \\ \frac{\omega|p|}{\rho^{p-1}} & \text{if } \omega \leq \rho \leq \omega^{\frac{1}{p-1}} \\ \frac{\omega|p|^{n+1}}{\rho^{p^n(p-1)}} & \text{if } \omega^{\frac{1}{p^{n-1}(p-1)}} \leq \rho \leq \omega^{\frac{1}{p^n(p-1)}}, \quad n \geq 1 \end{cases} \quad (9.14)$$

- iii) By the Small Radius Lemma 9.2, one finds

$$|g_0(T)|_\rho = \begin{cases} |g_0(T)|_\rho \leq \rho^{-1} & \text{if } 0 \leq \rho \leq \omega \\ \rho^{p-1}/|p| & \text{if } \omega \leq \rho \leq \omega^{\frac{1}{p-1}} \\ \rho^{p^n(p-1)}/|p|^{n+1} & \text{if } \omega^{\frac{1}{p^{n-1}(p-1)}} \leq \rho \leq \omega^{\frac{1}{p^n(p-1)}}, \quad n \geq 1 \end{cases} \quad (9.15)$$

Proof. To prove the theorem we have to study the equation (9.8) and all its iterates

$$p^n\text{-th equation :} \quad \sigma_{1,p^n}(Y) = A(1, p^n; T)Y . \quad (9.16)$$

Every function $A(1, p^n; T)$ is a polynomial and hence converge on all $K = \mathbb{Q}_p$. As already mentioned we will glue step by step the cocycles $Y_{p^n}(x, y)$, solution of the p^n th equation, defined by the expression (8.11). We will obtain a cocycle $Y_0(x, y)$ over $D^-(0, 1)$. Denote by M_{p^n} (resp. M_0) the σ_{1,p^n} -module (resp. the differential equation) defined in the basis \mathbf{e} by the equation (9.16) (resp. by the differential equation $Y' = g_0(T) \cdot Y$). We denote by r_n the largest radius such that M_{p^n} is σ_{1,p^n} -compatible over $D^-(0, r_n)$. To simplify the notations we set

$$R(p^n, \rho) := \liminf_s \left(|G_{[s]}(1, p^n; T)|_\rho / |[s]_q| \right)^{-1/s}, \quad (9.17)$$

where $G_{[s]}(1, p^n; T)$ have been defined in Section 8.3. Since $Y_0(T, 0) = \Gamma_p^0(T)$ converges exactly on the disk $D^-(0, \omega \cdot |p|^{1/p})$ (cf. [Dwo82]), the radius of convergence of $Y(T, c)$ is equal to $\omega \cdot |p|^{1/p}$, for all $c \in D^-(0, \omega \cdot |p|^{1/p})$, $c \in \Omega$. In particular this holds for every Dwork generic point t_ρ corresponding to $|\cdot|_\rho$. Hence, for all $n \geq 1$, we have

$$R(p^n, \rho) = \omega |p|^{1/p}, \quad \text{for all } \rho \leq \omega |p|^{1/p}. \quad (9.18)$$

Notice that, for all $n \geq 1$, the function $\rho \mapsto R(p^n, \rho)$ is limit of log-convex functions defined everywhere. So $R(p^n, \rho)$ is defined on the whole $\mathbb{R}_{\geq 0}$, and is log-concave and continuous on $\mathbb{R}_{\geq 0}$. Moreover one has

$$R(p^{n+1}, \rho) = R(p^n, \rho), \quad \text{for all } \rho \leq r_n. \quad (9.19)$$

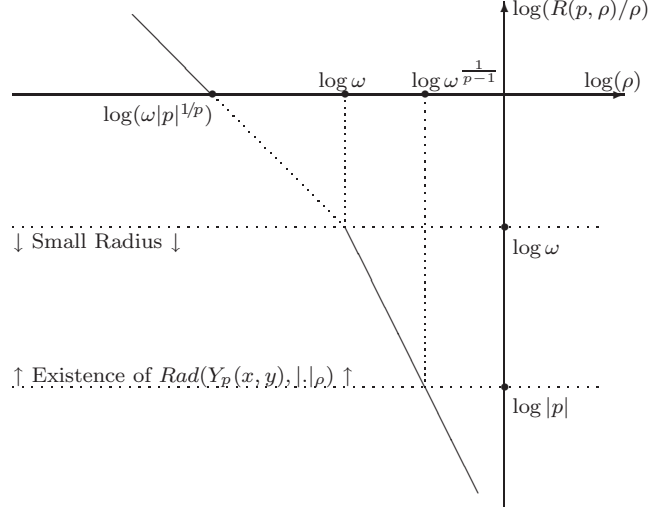
In other words if $\rho < r_n$ then both M_{p^n} and $M_{p^{n+1}}$ are σ_{1,p^n} -admissible and admits, by Theorem 5.27, the same analytic cocycle $Y(x, y)$ as solution. Hence, for $\rho < r_n$, both $R(p^n, \rho)$ and $R(p^{n+1}, \rho)$ represent the radius of the same analytic cocycle.

Step 1: Study of M_p . As already mentioned, we have $R(p, \rho) = \omega \cdot |p|^{1/p} > |p|$, for all $\rho \leq \omega \cdot |p|^{1/p}$. By Proposition 8.9, this proves that M_p is $\sigma_{1,p}$ -admissible over the disk $D^-(0, \omega |p|^{1/p})$, and hence we have $\text{Rad}(Y_1(x, y), \rho) = \rho$, for all $\rho \leq \omega |p|^{1/p}$.

LEMMA 9.4. *The radius $R(p, \rho)$ is equal to $\omega |p|^{1/p}$ for $\rho \leq \omega$, and is equal to $\omega |p|/\rho^{p-1}$ for $\rho \geq \omega$.*

Proof. Using Lemma 9.5 below, we applies the Small Radius Lemma 9.2, and we have $R(p, \rho) = \omega/\rho^{p-1}$ for all $\rho \geq \omega$. By the above discussion we have also $R(p, \rho) = \omega |p|^{1/p}$ for all $\rho \leq \omega |p|^{1/p}$. Now by continuity of $\rho \mapsto R(p, \rho)$, and by its log-concavity we must have $R(p, \rho) = \omega |p|^{1/p}$, for all $\rho \leq \omega$. The situation is expressed by the following picture in which we draw the log-graphic of the

function $\rho \mapsto R(p, \rho)/\rho$ i.e. the graphic of the function $\log(\rho) \mapsto \log(R(p, \rho)/\rho)$.



(9.20)

□

LEMMA 9.5. Let $G_{[1]}(1, p^n; T) := \frac{A(1, p^n; T) - 1}{p^n}$. For all $n \geq 1$ one has

$$|G_{[1]}(1, p^n; T)|_\rho = \frac{\rho^{\deg(G_{[1]}(1, p^n; T))}}{|p|^n} = \frac{\rho^{p^{n-1}(p-1)}}{|p|^n}, \quad \text{for all } \rho \geq 1. \quad (9.21)$$

Moreover for $n = 1$ one has

$$|G_{[1]}(1, p; T)|_\rho = \rho^{p-1}/|p|, \quad \text{for all } \rho \geq \omega. \quad (9.22)$$

Proof. The first equality results easily by writing explicitly $A(1, p^n; T)$ (cf. expression (9.10)). Indeed $A(1, p^n; T) - 1$ is a polynomial with coefficients (in \mathbb{Z}) having valuation smaller than 1. The assertion about $G_{[1]}(1, p; T)$ is proved as follows. Since the reduction of $A(1, p; T) - 1$ in $\mathbb{F}_p[T]$ is equal to $-T^{p-1}$, then $A(1, p; T) - 1 = -T^{p-1} + a_{p-2}T^{p-2} + \dots + a_0$, with $|a_i| \leq |p|$, for all $i = 0, \dots, p-2$. Hence $|A(1, p; T) - 1|_\rho = \max(\rho^{p-1}, |a_{p-2}|\rho^{p-2}, \dots, |a_0|)$, and $\rho^{p-1} \geq |p|\rho^i$ for all $i = 0, \dots, p-2$ if and only if $\rho \geq \omega = |p|^{1/(p-1)}$. □

COROLLARY 9.6. The following assertions hold:

- i) The largest disk $D^-(0, r_1)$ over which the $\sigma_{1,p}$ -module M_p is Taylor admissible has radius $r_1 = \omega^{1/(p-1)}$;
- ii) The cocycle $Y_p(x, y)$ attached to M_p by the expression (8.11), is solution of every equation of the family (9.16), and of the differential equation (9.13). One has

$$Y_p(x, y) = Y_{p^n}(x, y) = Y_0(x, y), \quad \text{for all } \rho \leq \omega^{1/(p-1)}. \quad (9.23)$$

- iii) One has

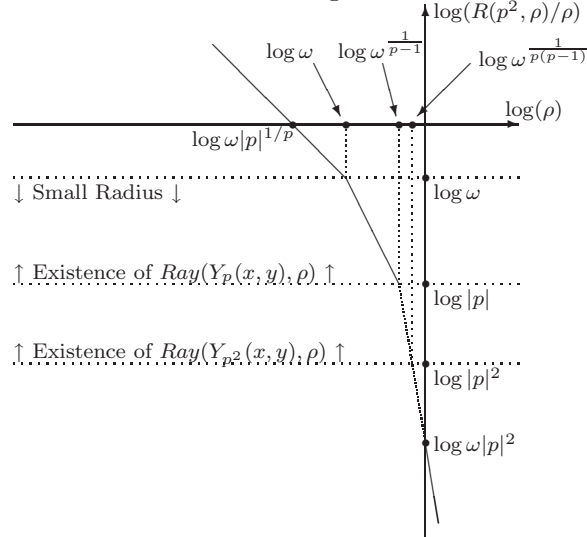
$$Rad_F(M_p, \mathbf{e}, \rho) = \begin{cases} \rho & \text{if } \rho \leq \omega|p|^{1/p} \\ \omega|p|^{1/p} & \text{if } \omega|p|^{1/p} \leq \rho \leq \omega \\ \omega|p|/\rho^{p-1} & \text{if } \omega|p|^{1/p} \leq \rho \leq \infty \end{cases} \quad (9.24)$$

and, for all $n \geq 1$, $Rad_F(M_p, \mathbf{e}, \rho) = Rad(Y_{p^n}(x, y), \rho) = Rad(Y_0, \rho)$, for all $\rho \leq \omega^{1/(p-1)}$. □

COROLLARY 9.7. One has $|g_0(T)|_\rho \leq \rho^{-1}$ if $\rho < \omega$, and $|g_0(T)|_\rho = \rho^{p-1}/|p|$ if $\omega \leq \rho \leq \omega^{\frac{1}{p-1}}$.

Proof. It follows immediately from the Small Radius Lemma applied to the differential equation $Y' = g_0(T)Y$ (cf. [CM02, Cor.6.7]). □

Step 2 : Study of M_{p^n} . By Lemma 9.5, applying the Small Radius Lemma to equations (9.21), one finds $Rad(M_2, \mathbf{e}, \rho) = \omega|p|^2/\rho^{p^2-p}$, for all $\rho \geq 1$. On the other hand we know that $Y_{p^2}(x, y) = Y_p(x, y)$ over $D^-(0, r_1) \times D^-(0, r_1)$, and hence that their radius coincide for $\rho \leq r_1 = \omega^{1/(p-1)}$ (cf. Corollary 9.6). We have then the following situation:



(9.25)

As before, we have the phenomenon that we know the radius $R(p^2, \rho)$ for $\rho \geq 1$, and for $\rho \leq \omega^{1/(p-1)}$, and again the radius $R(p^2, \rho)$ is forced by its continuity and its log-concavity to be equal to $R(p^2, \rho) = \omega|p|^2/\rho^{p^2-p}$ for all $\rho \geq \log \omega^{1/(p-1)}$. More precisely as above we have the following statements:

- i) $R(p^2, \rho) = \omega|p|^2/\rho^{p^2-p}$ for all $\rho \geq \log \omega^{1/(p-1)}$,
- ii) The module M_{p^2} is σ_{1,p^2} -admissible for all $\rho \leq r_2 := \omega^{\frac{1}{p(p-1)}}$,
- iii) $Rad(Y_p(x, y), \rho) = Rad(Y_{p^2}(x, y), \rho)$, for all $\rho \leq \omega^{1/(p-1)}$, and $Rad(Y_{p^2}, \rho) = \omega|p|^2/\rho^{p^2-p}$, for all $\omega^{1/(p-1)} \leq \rho \leq \omega^{1/p(p-1)}$,
- iv) $Rad(Y_0(x, y), \rho) = Rad(Y_{p^2}(x, y), \rho)$ for all $\rho \leq r_2$,
- v) $|g_0(T)|_\rho = \rho^{p^2-p}/|p|^2$ for all $\omega^{\frac{1}{p-1}} \leq \rho \leq \omega^{\frac{1}{p(p-1)}}$.

One checks now easily that this process can be iterated indefinitely. Continuing in this way, step by step, one computes completely the Radius $R(p^n, \rho)$ of each p^n th equation, one proves that M_{p^n} is Taylor admissible over $D^-(0, r_n)$, with $r_n = \omega^{1/p^{n-1}(p-1)}$. This gives the the Radius of $Rad(Y_0(x, y), \rho)$ for all $\rho < 1$. By the Small Radius Lemma we obtain the corresponding assertion about $|g_0(T)|_\rho$.

The fact that $g_0(T) \in \mathbb{Q}_p[[T]]$ follows by Corollary 8.16 since every function $A(1, p^n; T) - 1/p^n$ belongs to $\mathbb{Q}_p[[T]]$. This concludes the proof of Theorem 9.3. \square

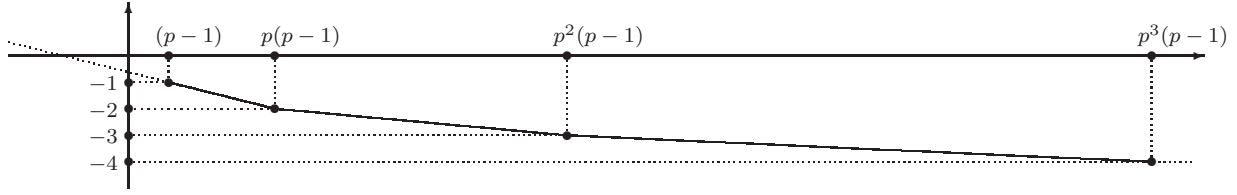
Define the Newton polygon of $g_0(T) := \sum_{n \geq 0} a_n T^n \in \mathbb{Q}_p[[T]]$ as the convex hull of the points $\{(n, v_p(a_n))\}_{n \geq 0} \cup \{0, +\infty\}$, where v_p is the p -adic valuation normalized by $v_p(p) = 1$.

COROLLARY 9.8. *The wedges of the Newton polygon of $g_0(T)$ having horizontal coordinate greater than $p-1$ are the points $\{(p^n(p-1), -n-1)\}_{n \geq 0}$. In particular*

- i) $v_p(a_{p^n(p-1)}) = -n-1$, for all $n \geq 0$,
- ii) Moreover for all $k \geq 0$ one has

$$v_p(a_k) \geq \begin{cases} 0 & \text{if } 0 \leq k \leq p-2, \\ -n-1 & \text{if } p^n(p-1) \leq k \leq p^{n+1}(p-1), \quad n \geq 0. \end{cases} \quad (9.26)$$

as illustrated in the following picture:



□

9.3 Applications to Kubota-Leopoldt's p -adic L -functions

We preserve the assumption $p \neq 2$. It has been known since Y. Morita [Mor75] and J. Diamond [Dia77] (see also [Rob00, p.376]) that the composite of \log with Γ_p admits the following Taylor expansion for $|T| \leq |p|$:

$$\log(\Gamma_p^0(T)) = \lambda_0 T + \sum_{m \geq 1} L_p(1+2m, \omega_p^{2m}) \frac{T^{1+2m}}{1+2m} \quad (9.27)$$

where here ω_p is the Teichmüller Dirichlet character corresponding to the prime p , and where $L_p(1+2m, \omega_p^{2m})$ is the value at $s = 1+2m$ of the p -adic Kubota-Leopoldt's L -function corresponding to the character ω_p^{2m} . The constant λ_0 is the constant coefficient appearing in the Taylor expansion of $\zeta_p(s)$ at $s = 1$: $\zeta_p(s) = \sum_{n \geq -1} \lambda_n (s-1)^n$. We notice now that

$$g_0(T) = \frac{d}{dT}(\log(\Gamma_p^0(T))) = \lambda_0 + \sum_{m \geq 1} L_p(1+2m, \omega_p^{2m}) T^{2m}. \quad (9.28)$$

This function has been studied in the previous section and is intimately related to the Radius of convergence of the differential equation $\Gamma_p^0(T)' = g_0(T)\Gamma_p^0(T)$. The Newton polygon of $g_0(T)$ has been computed in Corollary 9.8. As a direct consequence we have the following estimate on the values of the L -functions appearing in (9.28):

COROLLARY 9.9. *For all $m \geq 1$ one has*

$$v_p(L_p(1+p^m(p-1), \omega_p^{p^m(p-1)})) = v_p(\zeta_p(1+p^m(p-1))) = -m-1, \quad (9.29)$$

and moreover

$$v_p(L_p(1+2m, \omega_p^{2m})) \leq \begin{cases} 0 & \text{if } 0 \leq 2m \leq (p-1) \\ -n-1 & \text{if } p^n(p-1) \leq 2m \leq p^{n+1}(p-1), \quad n \geq 0. \end{cases}$$

Notice that this agrees with the fact that $\zeta_p(s)$ has a pole at $s = 1$.

9.3.1 We now use the fact that the function $\Gamma_p^0(T)$ is simultaneously solution of every difference equation $\sigma_{1,p^n}(Y) = A(1, p^n; T)Y$ studied in the previous section, and of the differential equation $\Gamma_p^0(T)' = g_0(T)\Gamma_p^0(T)$. We obtain a family of congruences satisfied by the values of the L -functions appearing in the Taylor expansion of $\log(\Gamma_p^0(T))$.

For all integers $\ell, k \geq 1$, we set

$$S_\ell(k) := \sum_{i=1, (i,p)=1}^{k-1} \frac{1}{i^\ell}. \quad (9.30)$$

This and similar sums have been studied by several authors modulo powers of p [Dic52, pp. 95-103]. A result of L. Washington [Was98] expresses it as sum of values at certain positive integers of some Kubota-Leopoldt's p -adic L -functions. Similar expressions have been found by D. Barsky [Bar83].

The following result provides analogous relations in the style of [Was98]. The proof of these new congruences seems more conceptual with respect to older methods because it is a direct consequence of Theorem 5.18 which express the Gamma function as simultaneous solution of the differential equation (9.13) and the difference equations (9.9).

COROLLARY 9.10. *For all $n \geq 1$ the following holds:*

i) *For $\ell = 0$ one has*

$$\log(\Gamma_p(p^n)) = p^n \lambda_0 + \sum_{m \geq 1} p^{n(1+2m)} \frac{L_p(1+2m, \omega_p^{2m})}{1+2m} \quad (\ell = 0); \quad (9.31)$$

ii) *For $\ell = 1$ one has*

$$S_1(p^n) := \sum_{i=1, (i,p)=1}^{p^n-1} \frac{1}{i} = \sum_{m \geq 1} p^{2mn} \cdot L_p(1+2m, \omega_p^{2m}) = g_0(p^n) - g_0(0). \quad (9.32)$$

In particular we recover the relation $\lim_{n \rightarrow \infty} p^{-n} \sum_{i=0, (i,p)=1}^{p^n-1} i^{-1} = 0$ because $g'_0(0) = 0$.

iii) *For all $\ell \geq 2$ one has*

$$\frac{(-1)^{\ell-1}}{\ell} \cdot S_\ell(p^n) = \sum_{m \geq \ell/2} \binom{1+2m}{\ell} p^{n(1+2m-\ell)} \cdot \frac{L_p(1+2m, \omega_p^{2m})}{1+2m} \quad (\ell \geq 2).$$

Proof : As formal power series $\Gamma_p^0(T)$ can be written as

$$\Gamma_p^0(T) = \exp\left(\lambda_0 T + \sum_{m \geq 1} L_p(1+2m, \omega_p^{2m}) \cdot \frac{T^{1+2m}}{1+2m}\right) \quad (9.33)$$

the functional equation gives then

$$\Gamma_p^0(T + p^n) / \Gamma_p^0(T) = A(1, p^n; T) = - \prod_{\substack{i=1, \\ (i,p)=1}}^{p^n-1} (T + i) \quad (9.34)$$

On the left hand side one finds

$$\Gamma_p^0(T + p^n) / \Gamma_p^0(T) = \exp\left(\lambda_0 p^n + \sum_{m \geq 1} L_p(1+2m, \omega_p^{2m}) \cdot \frac{(T + p^n)^{1+2m} - T^{1+2m}}{1+2m}\right) \quad (9.35)$$

Now we compute the argument of the exponential. To simplify the notations let $a_{1+2m} := L_p(1+2m, \omega_p^{2m}) / (1+2m)$, then

$$\begin{aligned} \sum_{m \geq 1} a_{1+2m} \cdot ((T + p^n)^{1+2m} - T^{1+2m}) &= \sum_{m \geq 1} a_{1+2m} \cdot \sum_{k=1}^{1+2m} \binom{1+2m}{k} p^{nk} T^{1+2m-k} \\ &= \sum_{\ell \geq 0} \left(\sum_{m \geq \max(1, \ell/2)} \binom{1+2m}{\ell} p^{n(1+2m-\ell)} \cdot a_{1+2m} \right) T^\ell \end{aligned} \quad (9.36)$$

Now taking log of both sides of (9.34) one finds

$$\lambda_0 p^n + \sum_{\ell \geq 0} \left(\sum_{m \geq \max(\ell/2, 1)} \binom{1+2m}{\ell} p^{n(1+2m-\ell)} \cdot a_{1+2m} \right) T^\ell = \log\left(- \prod_{\substack{i=1, \\ (i,p)=1}}^{p^n-1} (T + i)\right). \quad (9.37)$$

Write then $-\prod_{i=1, (i,p)=1}^{p^n-1} (T+i) = \Gamma_p(p^n) \cdot \prod_{i=1, (i,p)=1}^{p^n-1} (1 + \frac{T}{i})$. Since $|\Gamma_p(p^n) - 1| \leq |p|$, then $\log(\Gamma_p(p^n))$ has a meaning. Then

$$\begin{aligned} \log\left(-\prod_{\substack{i=1, \\ (i,p)=1}}^{p^n-1} (T+i)\right) &= \log(\Gamma_p(p^n)) + \sum_{i=1, (i,p)=1}^{p^n-1} \log\left(1 + \frac{T}{i}\right) \\ &= \log(\Gamma_p(p^n)) + \sum_{\ell \geq 1} \frac{(-1)^{\ell-1}}{\ell} S_\ell(p^n) T^\ell \end{aligned}$$

This proves the corollary. \square

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14, 2008